香港電腦奧林匹克競賽
Hong Kong Olympiad in Informatics

Flow and Graph Matching (I)

Ethen Yuen {ethening}

2026-04-18

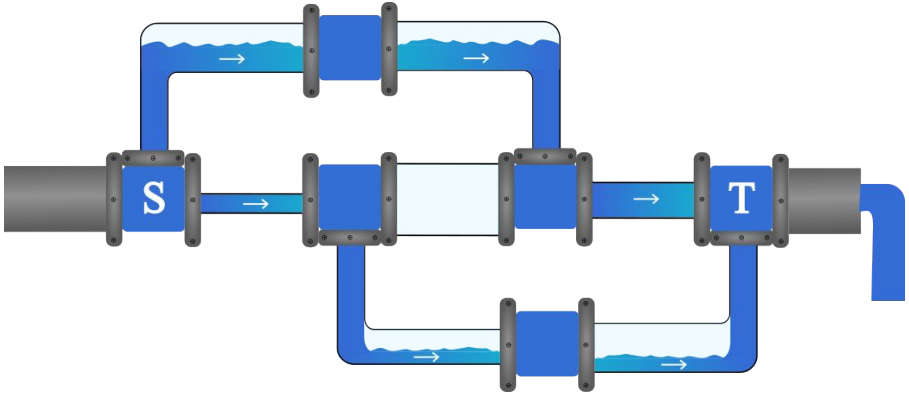
Agenda

- Maximum Flow
  - Ford-Fulkerson Algorithm
  - Modeling Technique
- Minimum Cut
  - Max-Flow Min-Cut Theorem
  - Set Partitioning Model and “Optimistic Thinking”
- *Team* Problem Set

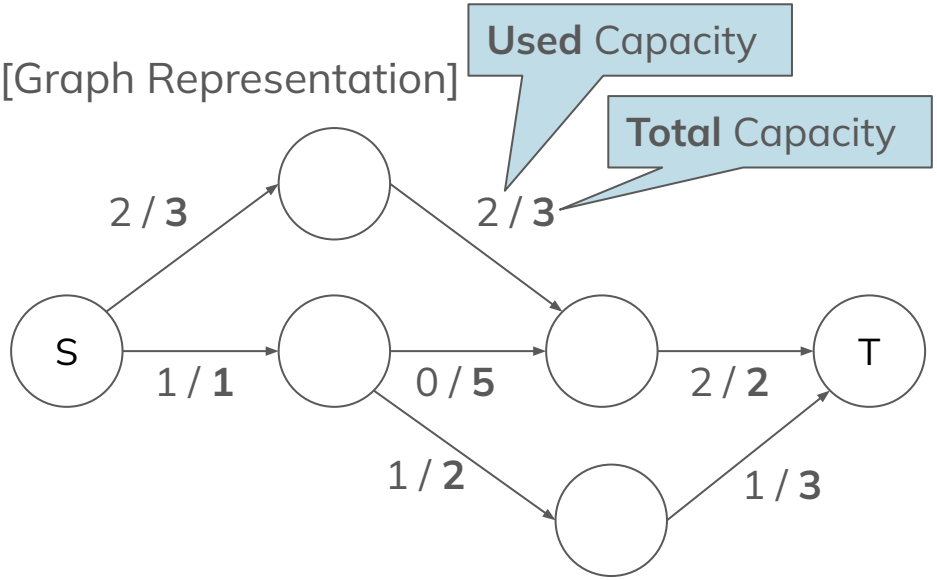
Maximum Flow

Maximum Flow Problem

[Flow Network in Real Life]

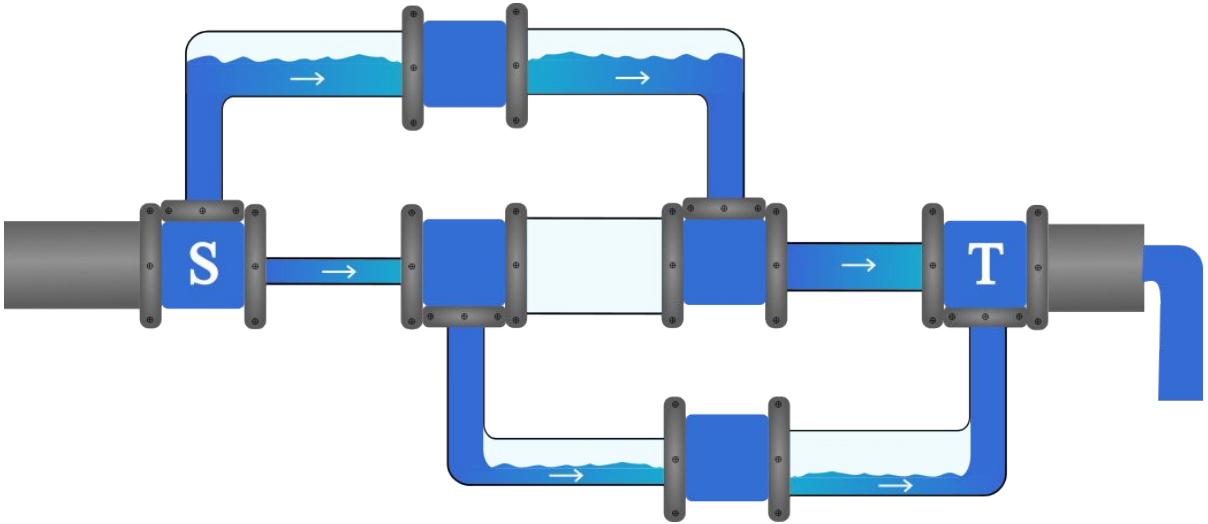


[Graph Representation]



Maximum Flow Problem

There are some nodes in a flow network, connected with **unidirectional** pipes. Each pipe has a **maximum capacity**. Find the maximum amount of flow we can send from the **source S** to the **sink T** per unit time.



Maximum Flow Problem

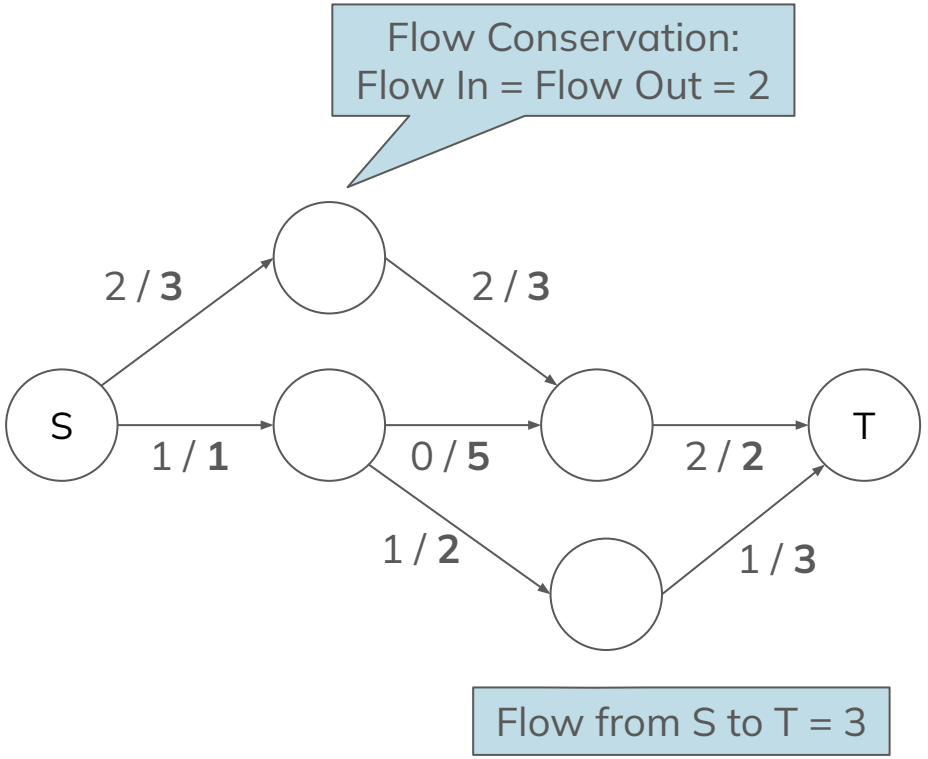
Flow Conservation: For all vertices except S and T, flow in = flow out.

We aim to maximize the **flow from S to T**:

(flow out for S) - (flow in for S)

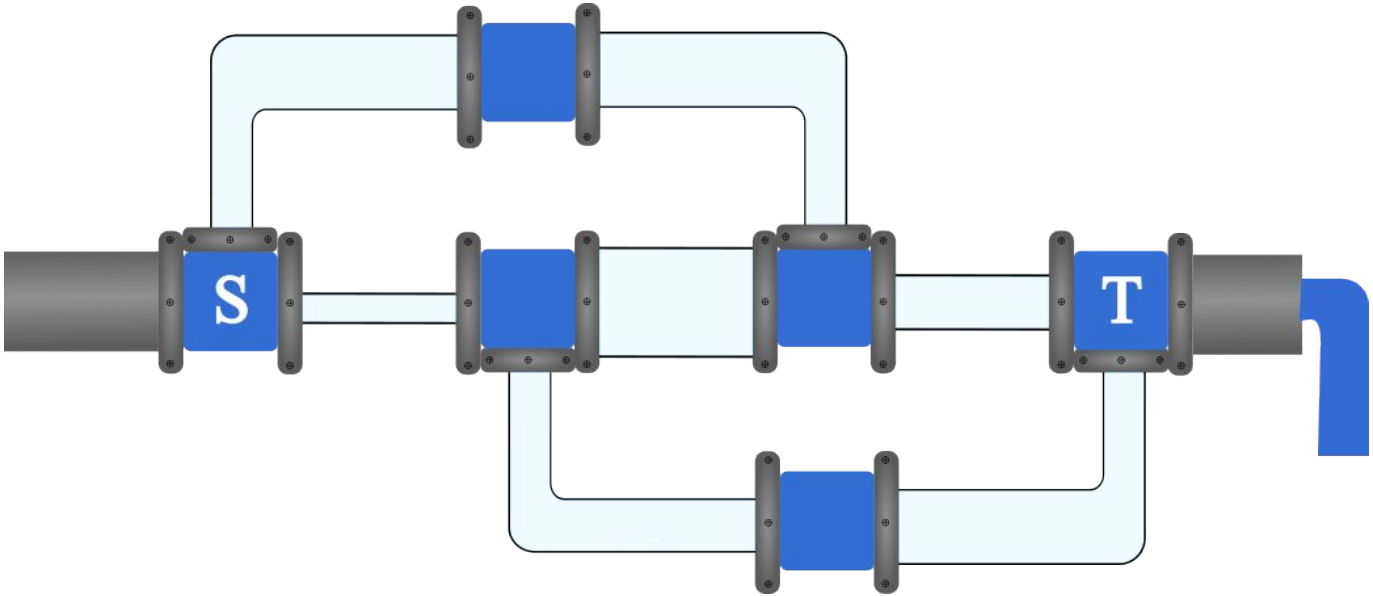
Or equivalently:

(flow in for T) - (flow out for T)



Ford–Fulkerson Algorithm – Intuition

You are asked to solve this problem in real life. What would you do?



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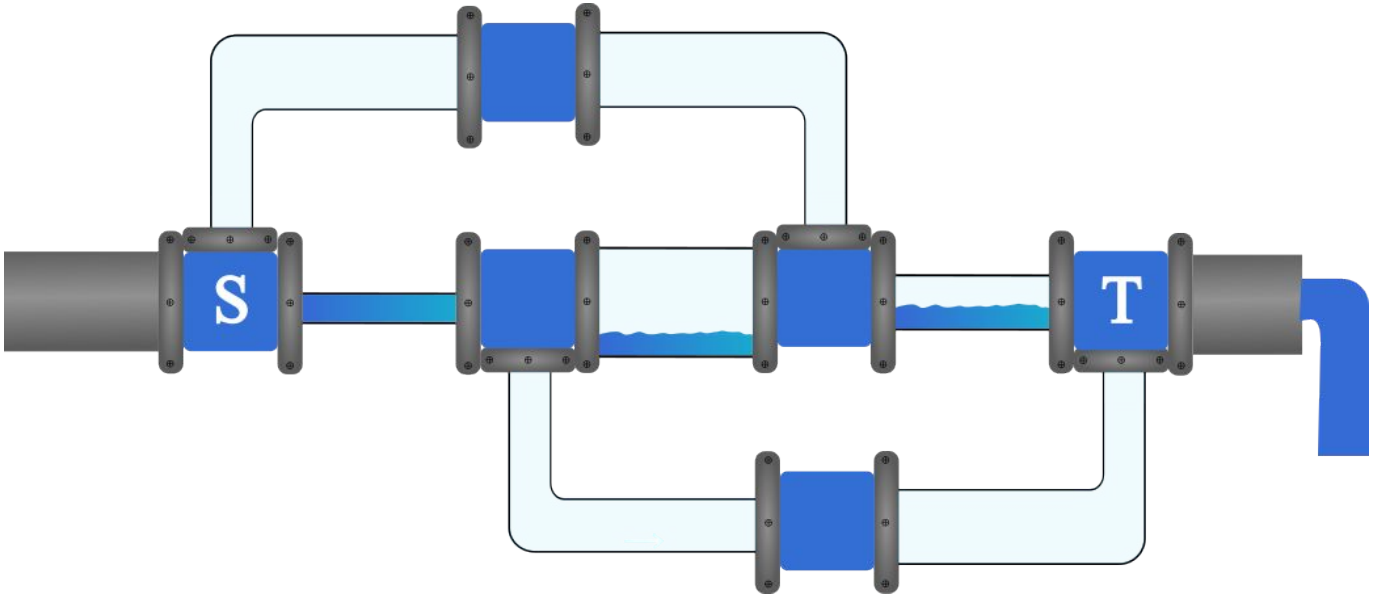

1. Turn on the tap

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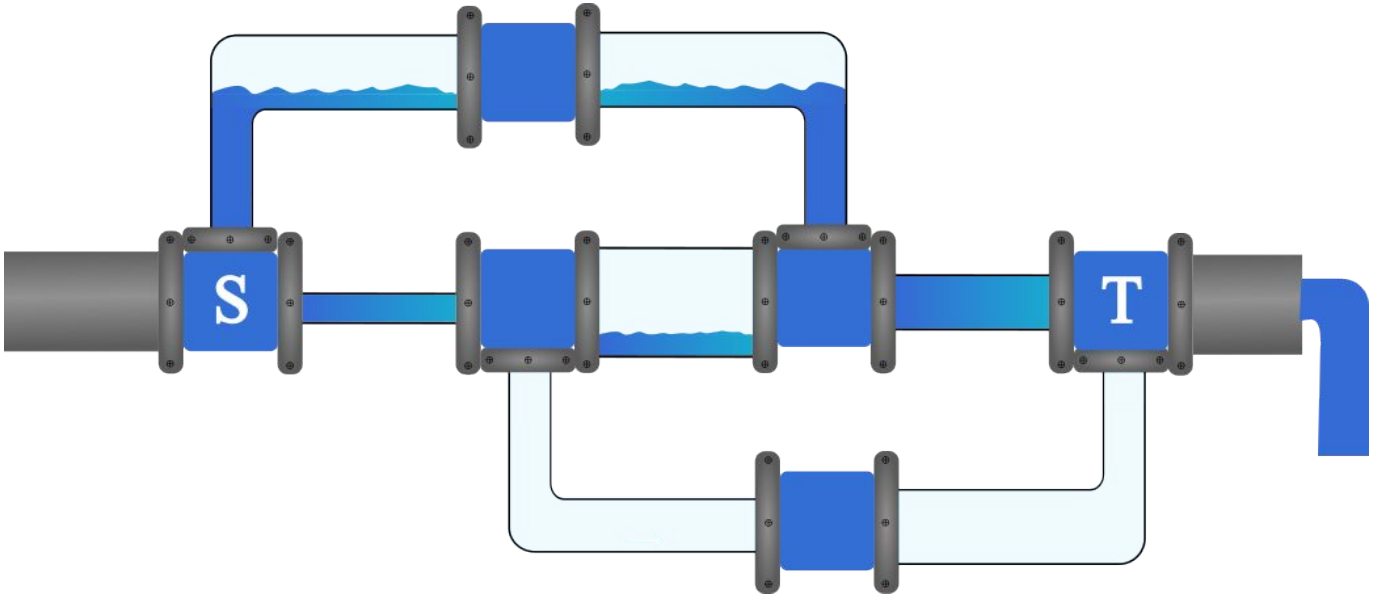

1. Turn on the tap
2. A bit more...

Ford–Fulkerson Algorithm – Intuition

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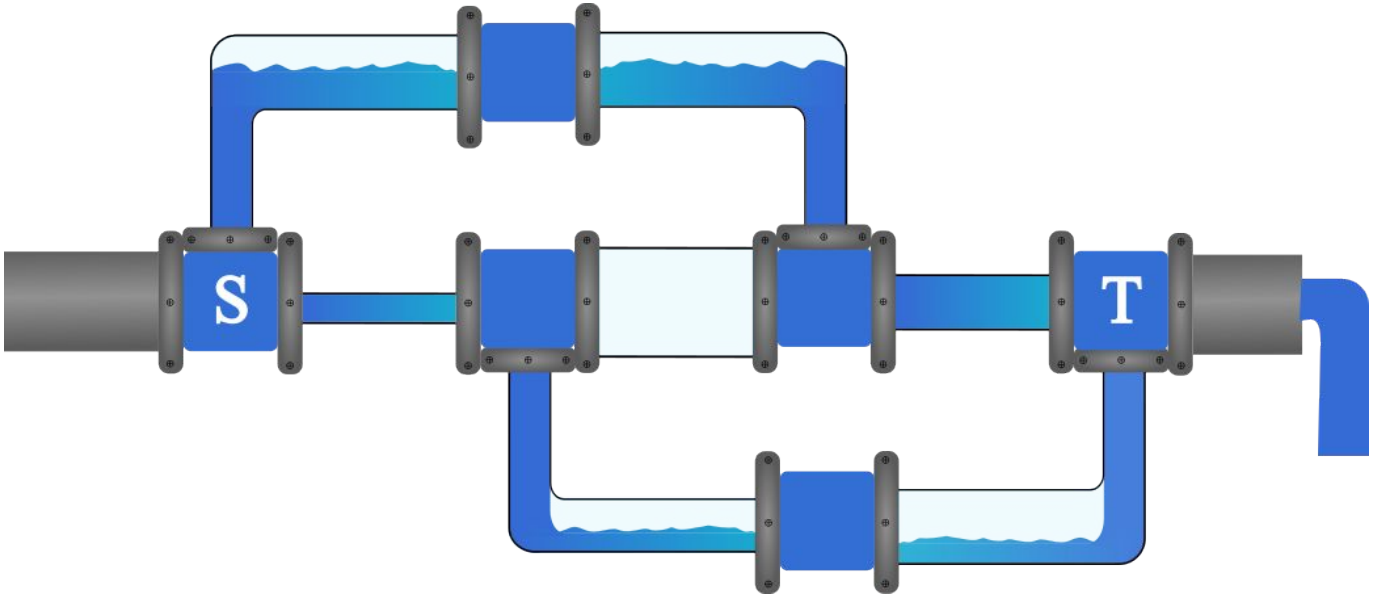

1. Turn on the tap
2. A bit more...
3. A bit more...

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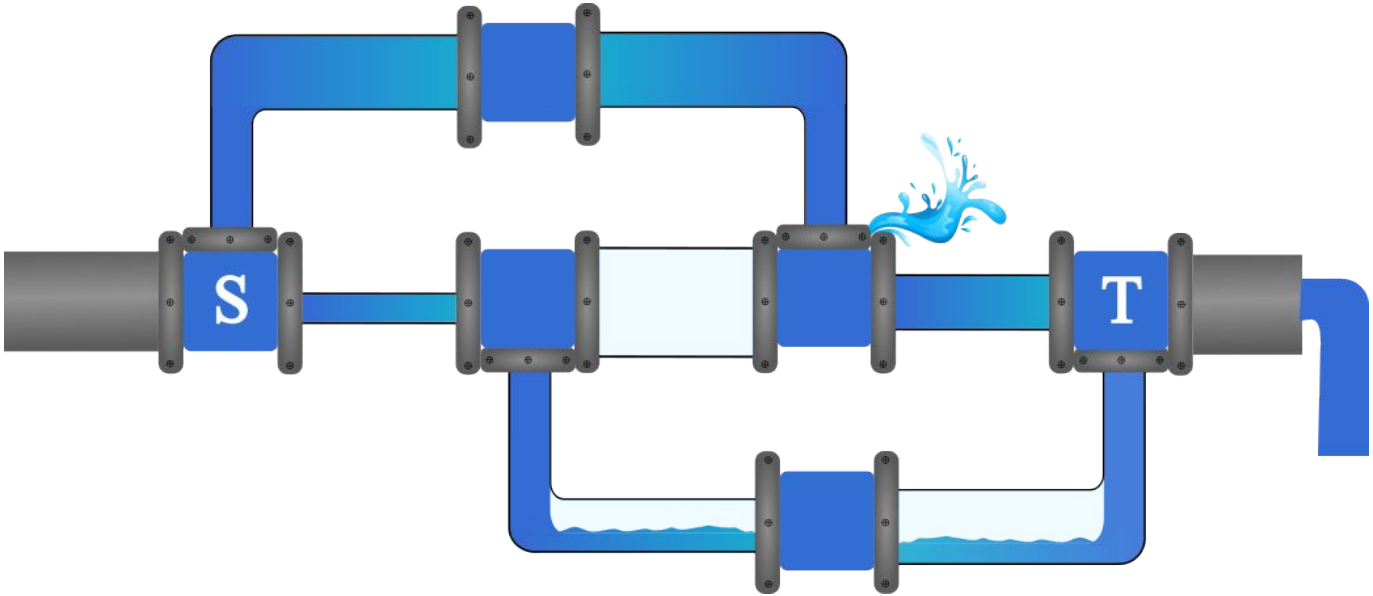

1. Turn on the tap
2. A bit more...
3. A bit more...

↑ Maximum Flow

4. A bit more...

No more flow to T

Ford–Fulkerson Algorithm – Intuition

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    Else
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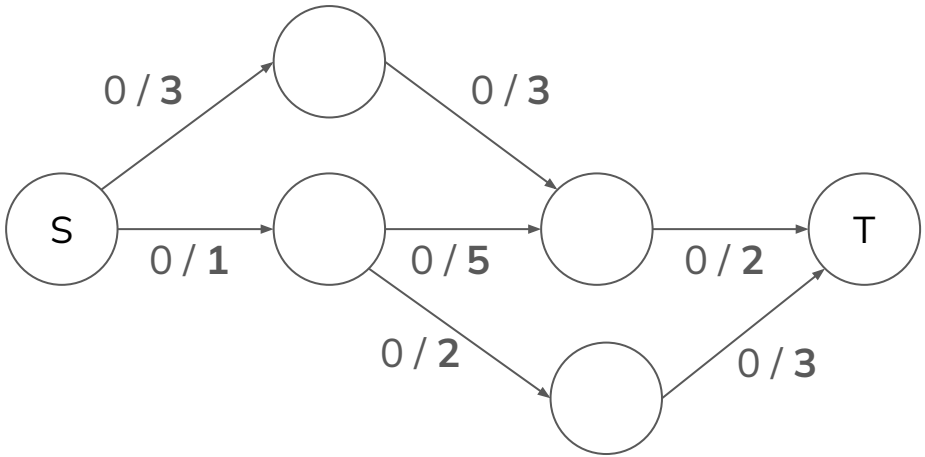
```



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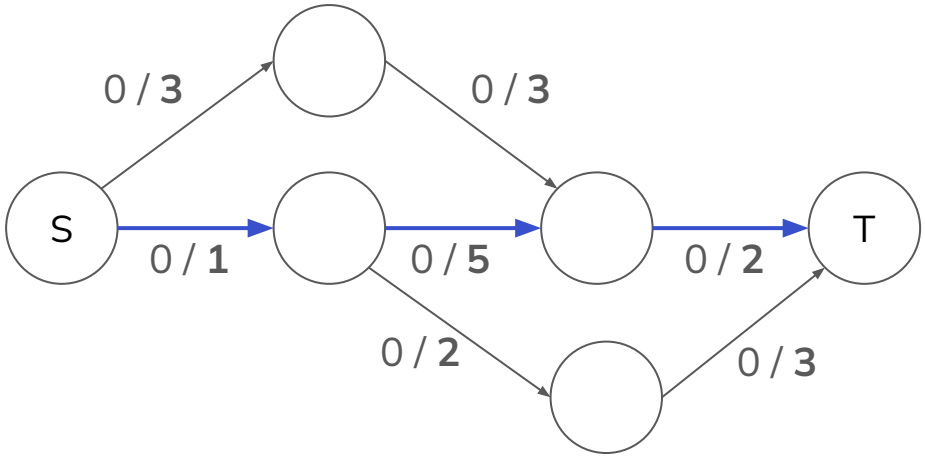

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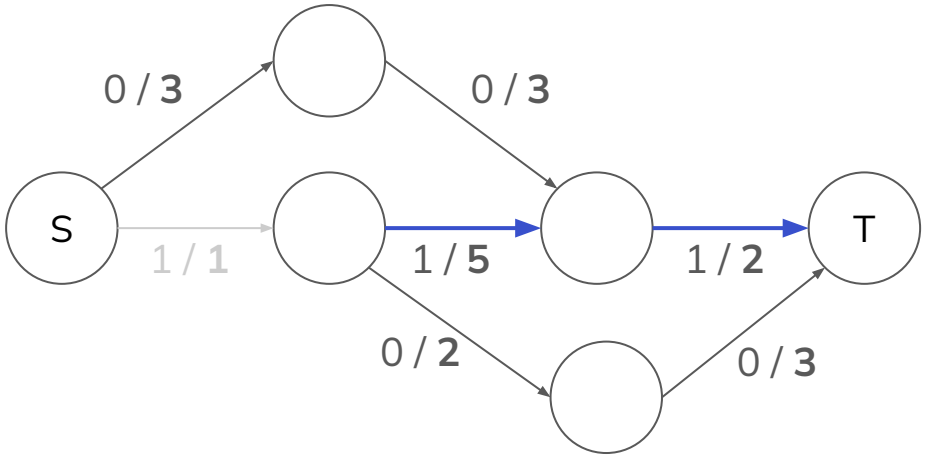

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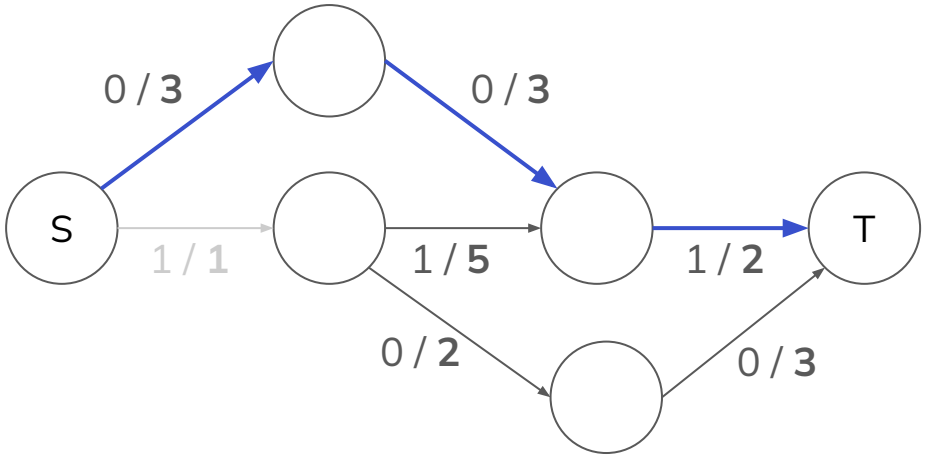

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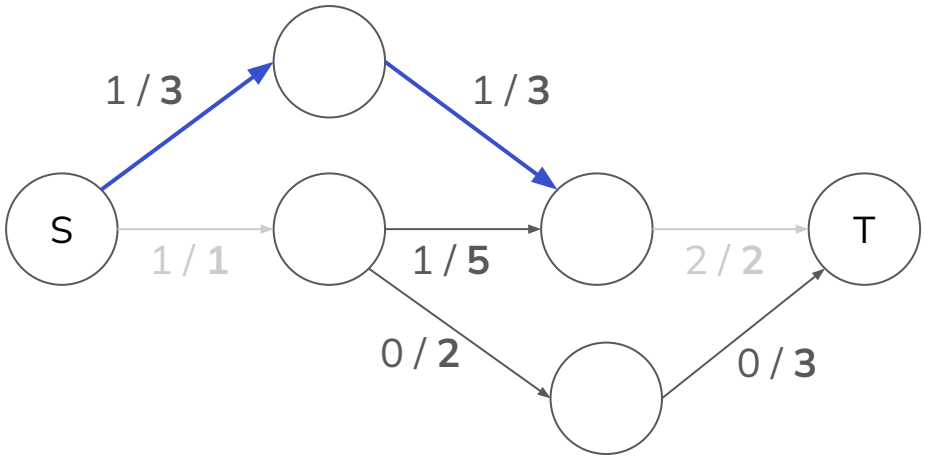

Ford–Fulkerson Algorithm – Intuition

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    Else
        Break out of loop
}
```
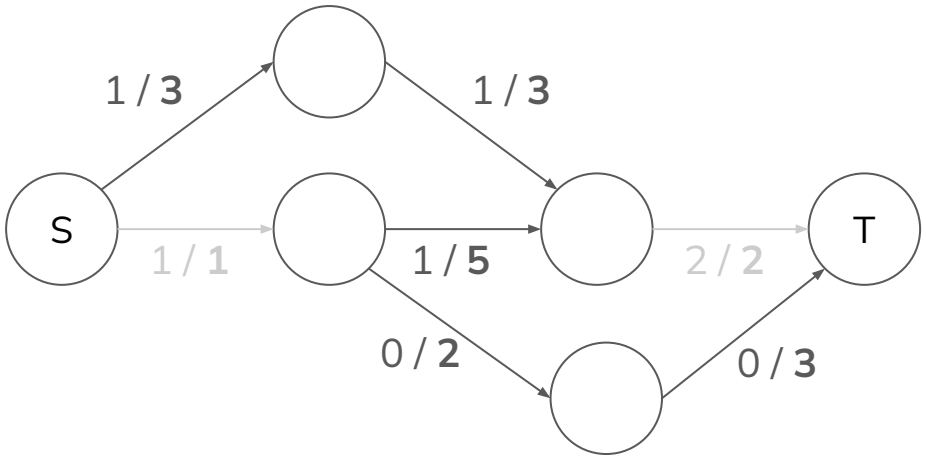
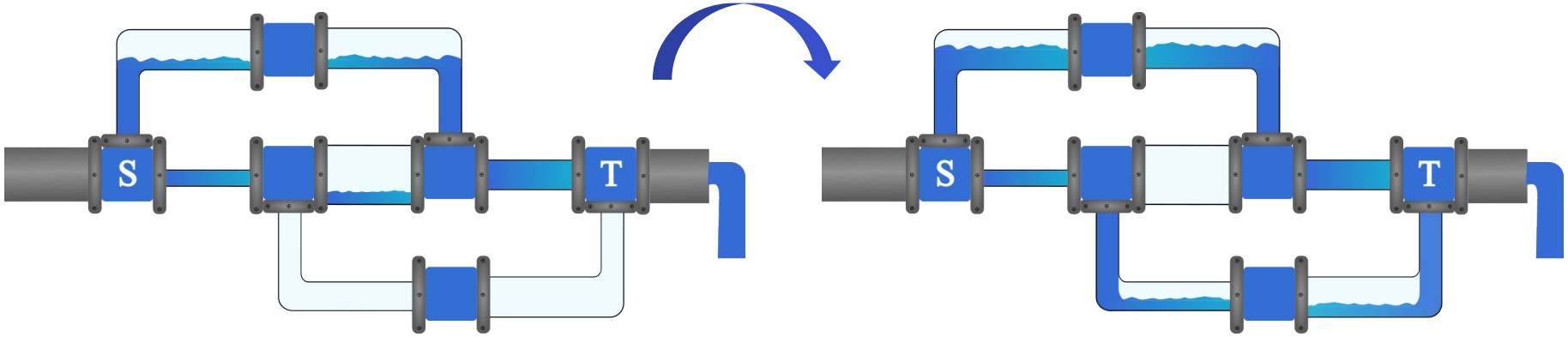

No more paths
Max Flow = 2?



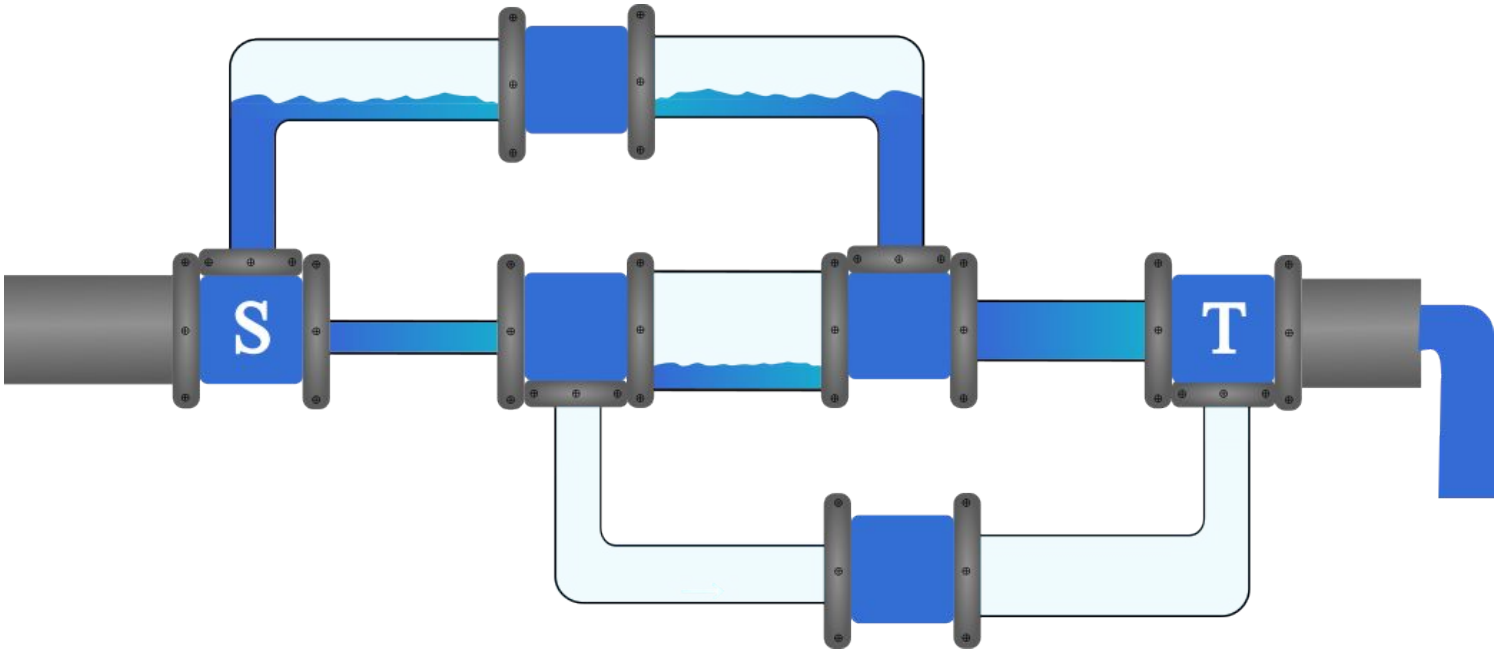
Ford–Fulkerson Algorithm – Intuition

Notice that in our “real life” example, the water will “magically” flow correctly when the flow increases from 2 to 3:



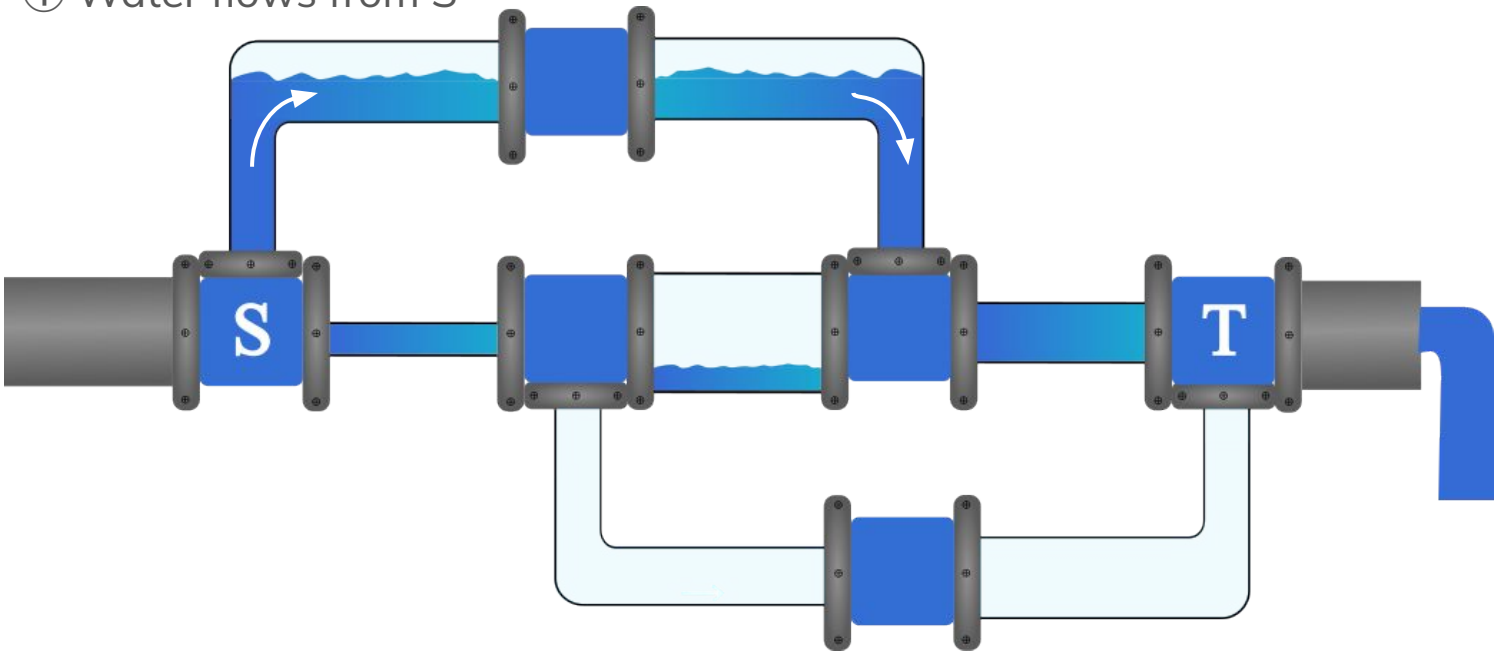
How did that happen?

Ford–Fulkerson Algorithm – Intuition

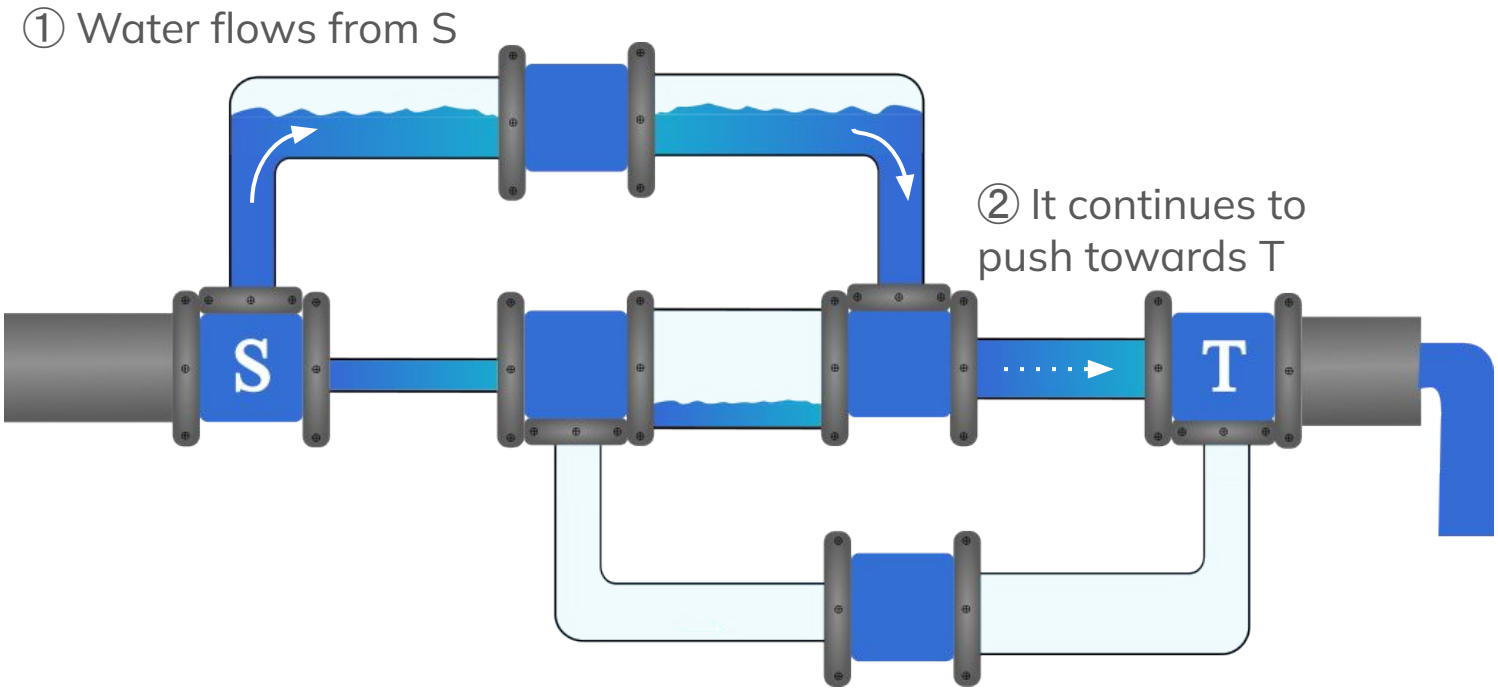


Ford–Fulkerson Algorithm – Intuition

① Water flows from S



Ford–Fulkerson Algorithm – Intuition



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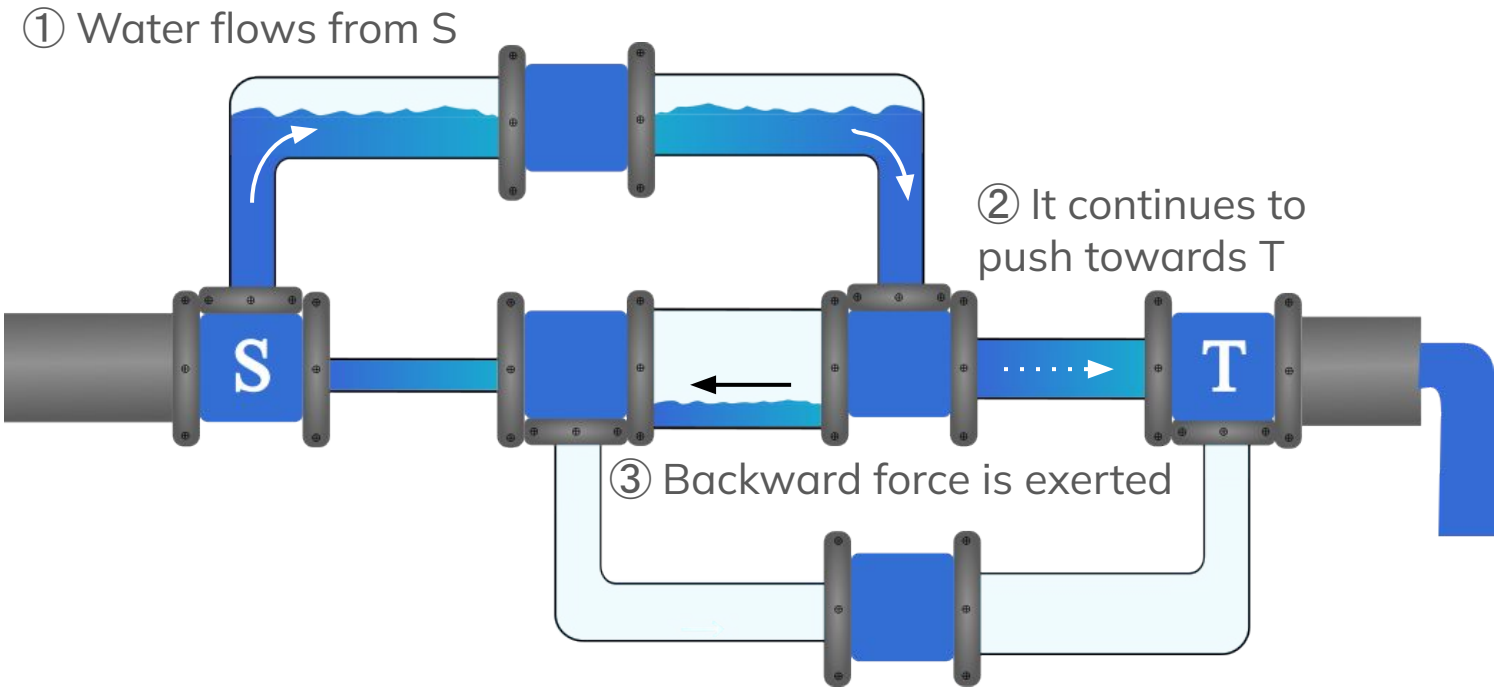

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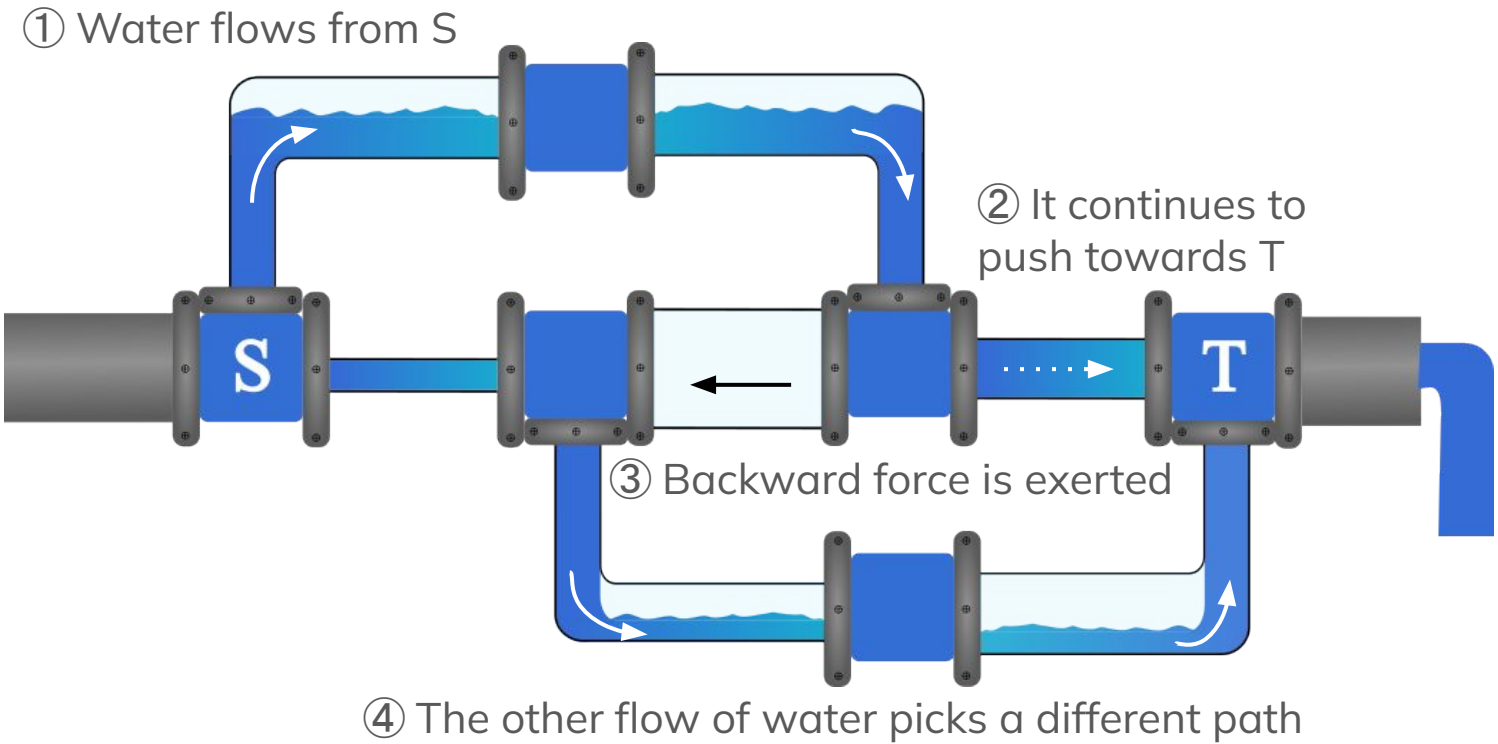
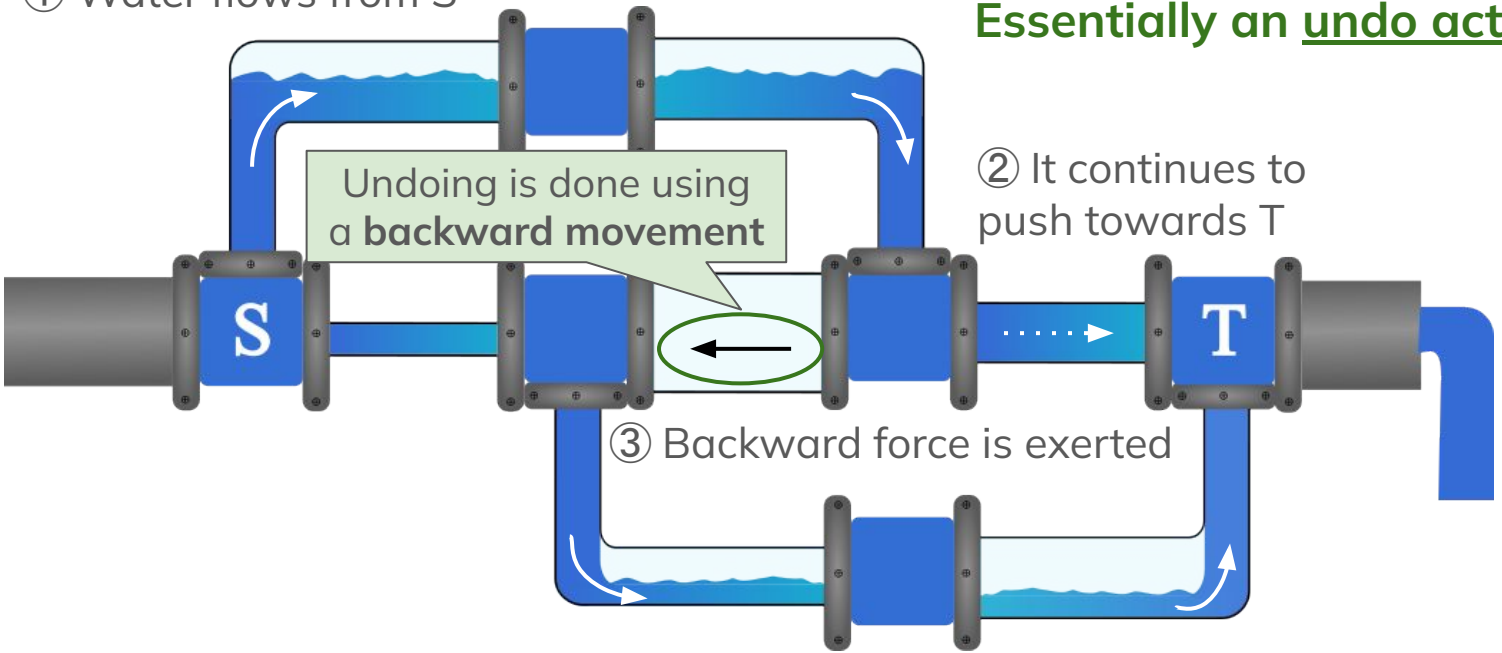


Ford–Fulkerson Algorithm – Intuition

① Water flows from S

Essentially an undo action!



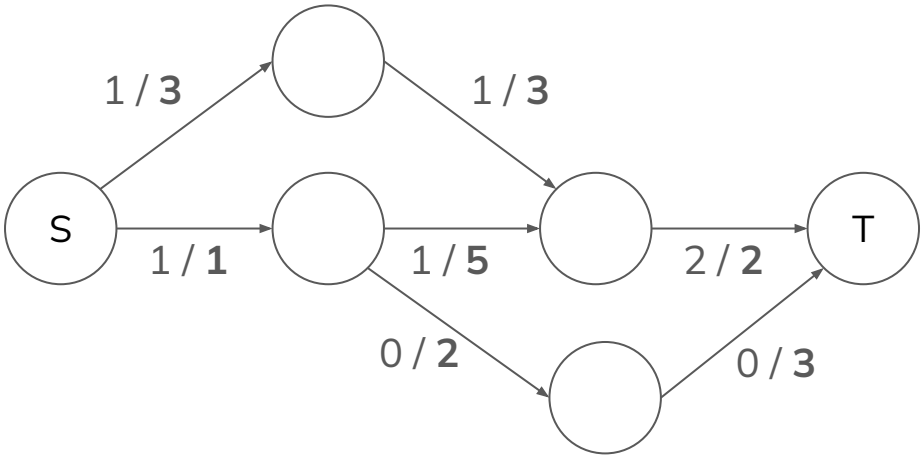
④ The other flow of water picks a different path

Ford–Fulkerson Algorithm – Implementation

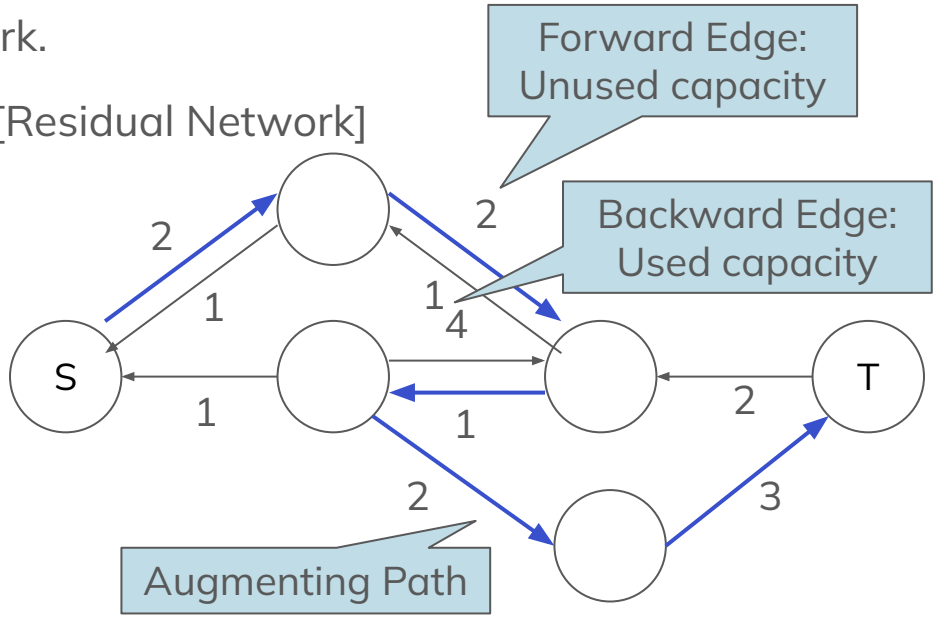
Residual Network: The current state of the flow network together with the backward edges for “undoing” actions.

Augmenting Path: Path on the residual network.

[Flow Network]



[Residual Network]



Ford–Fulkerson Algorithm – Implementation

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REPEAT {
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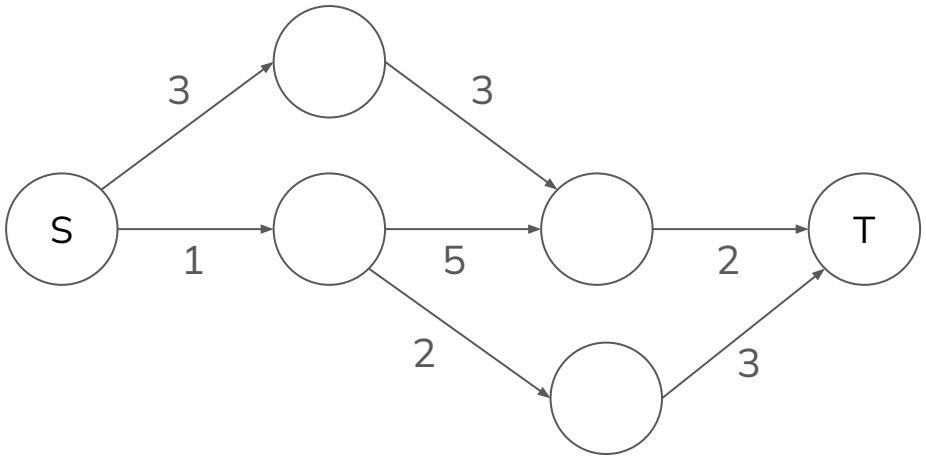
        Break out of loop
}
```



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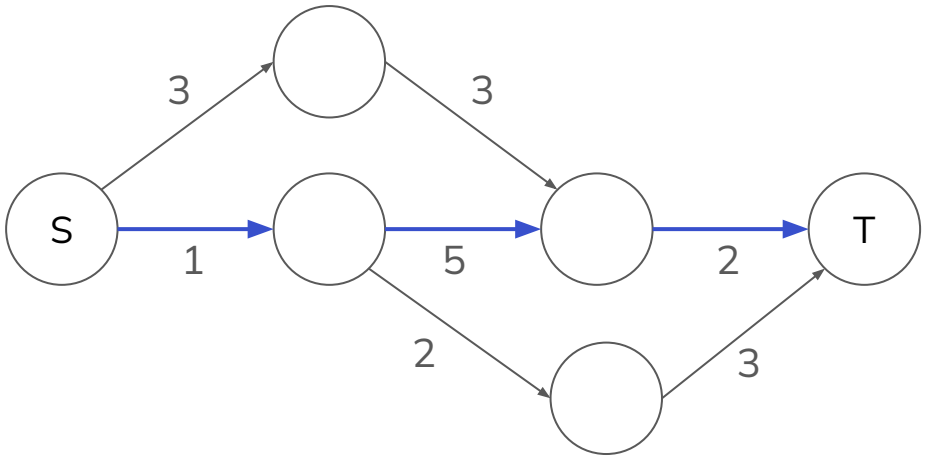

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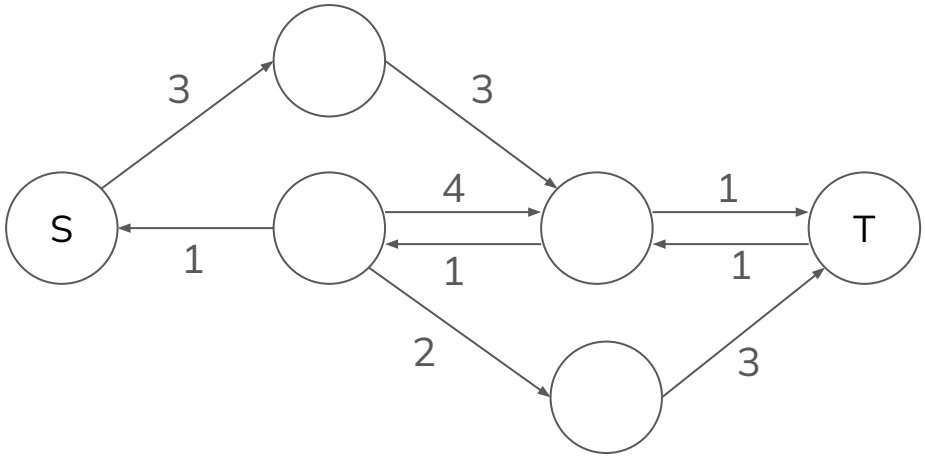

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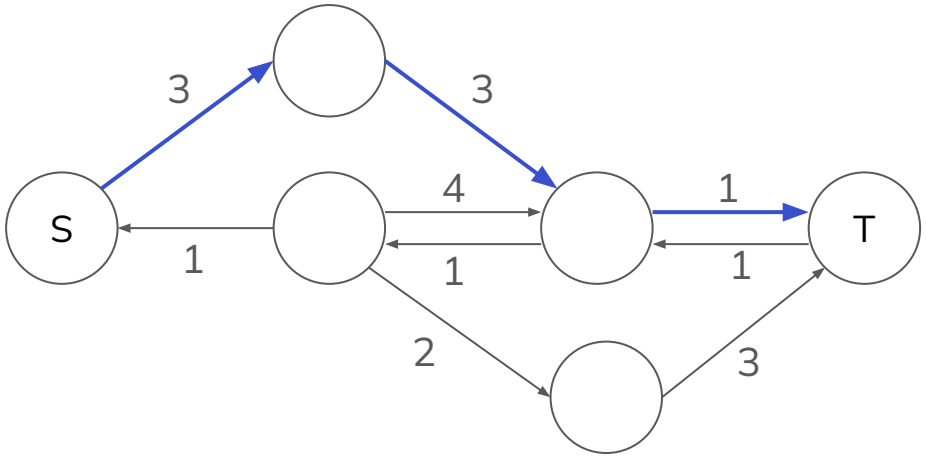

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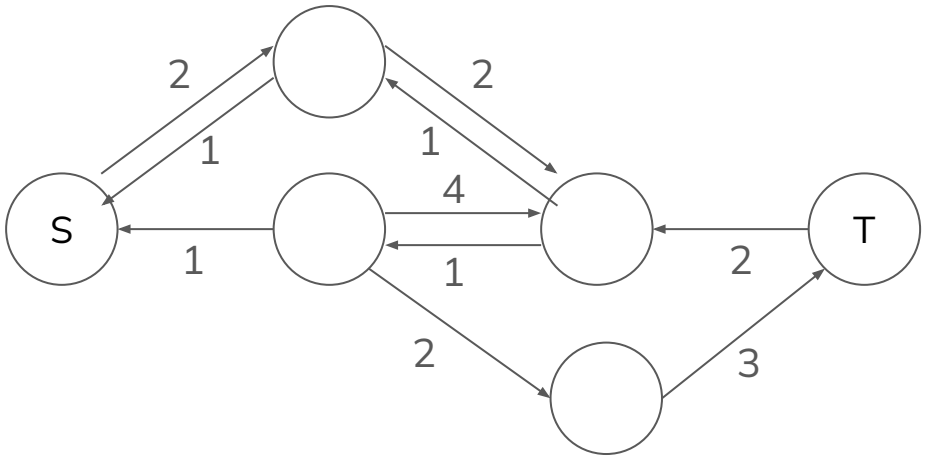

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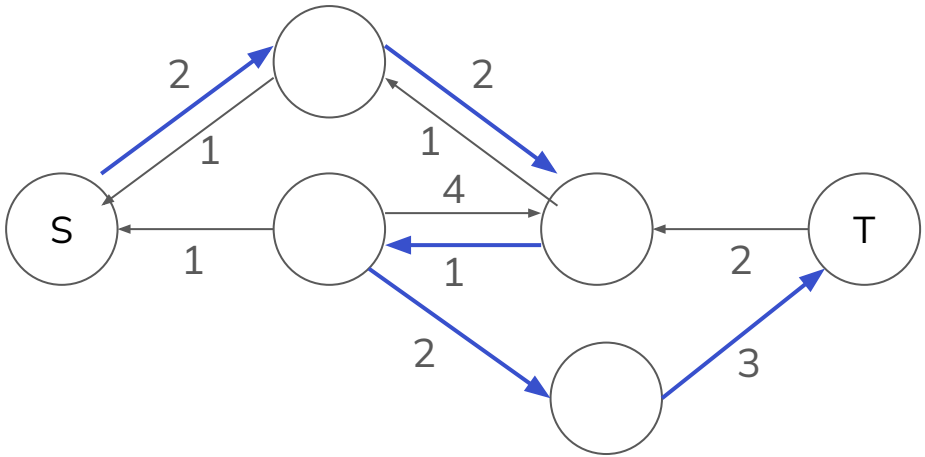

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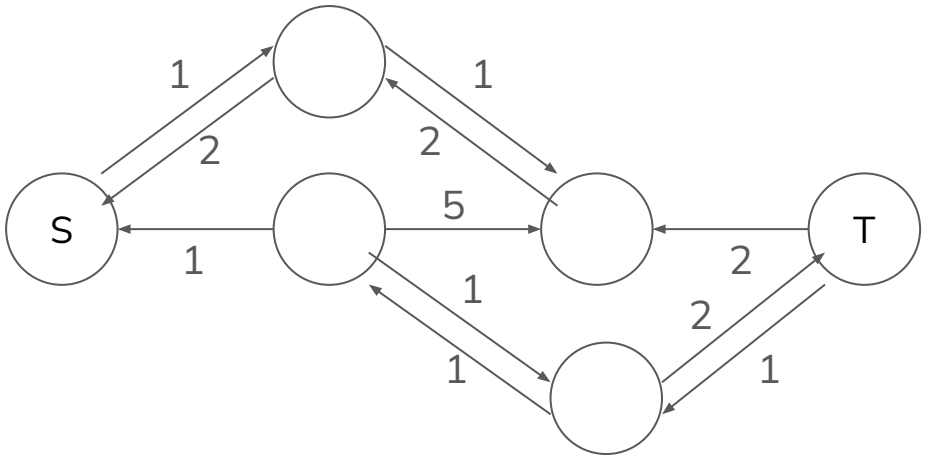

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```
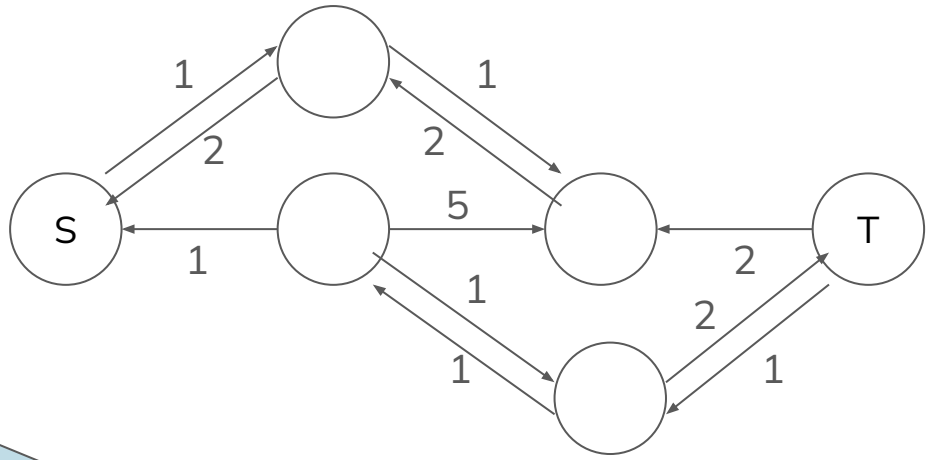

No more paths
Max Flow = 3



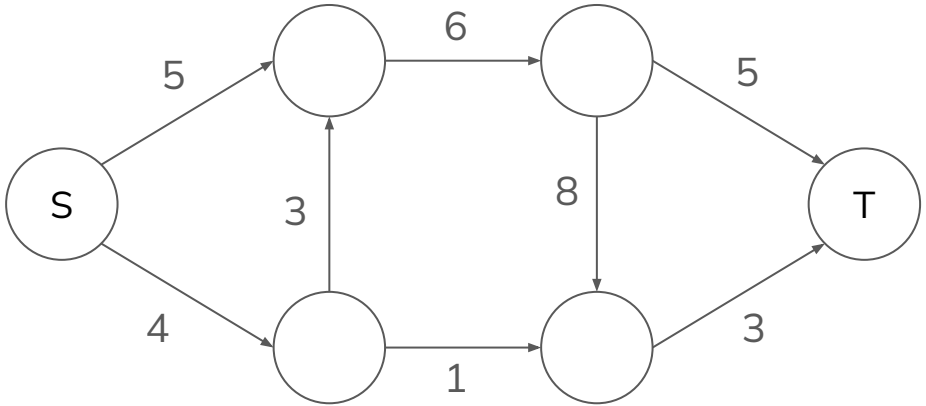
Time Complexity: **O(EF)**

Number of edges

Maximum flow

Quick Check

🔤 Consider the following flow network. Find the maximum flow from S to T.



(Answer: 7)

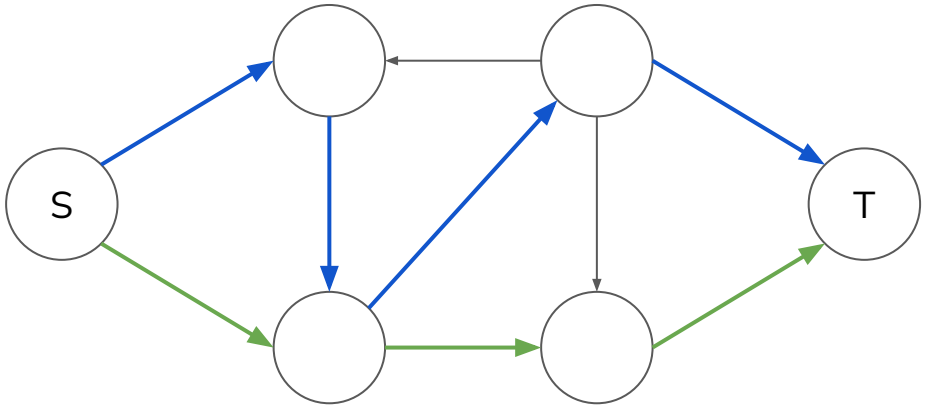
Other Maximum Flow Algorithms

- There exists other faster flow algorithms (not within the scope of IOI).

Year	Method	Time Complexity
1955	Ford–Fulkerson algorithm	O(EF)
1970	Dinic's algorithm	$O(V^2E)$
1983	Dinic's algorithm + dynamic trees	O(VE log V)
1988	Push–relabel algorithm + dynamic trees	$O(VE \log(V^2/E))$
…		
2022	Chen, Kyng, Liu, Peng, Gutenberg and Sachdeva's algorithm	$O(E^{1+o(1)} \log F)$ *** Almost linear**

Solving Graph Problems with Flow

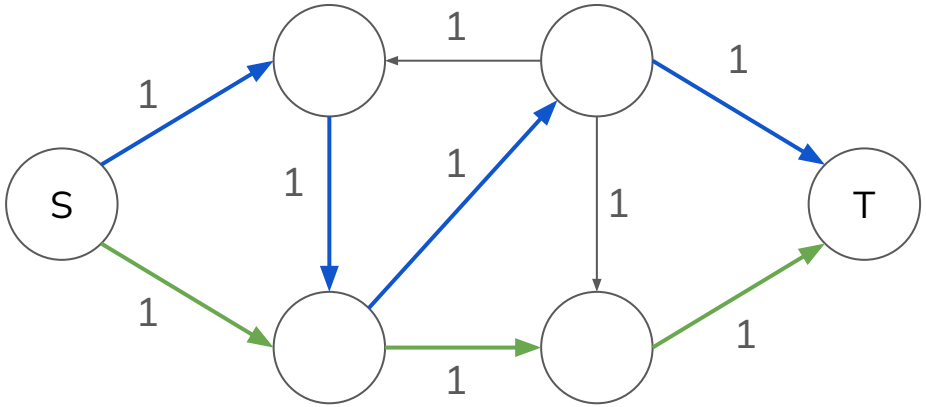
Edge-disjoint paths: Find the maximum number of **edge-disjoint paths** from the **source** to the **sink**, i.e. each edge appears in **at most one path**.



Solving Graph Problems with Flow

Edge-disjoint paths: Find the maximum number of **edge-disjoint paths** from the **source** to the **sink**, i.e. each edge appears in **at most one path**.

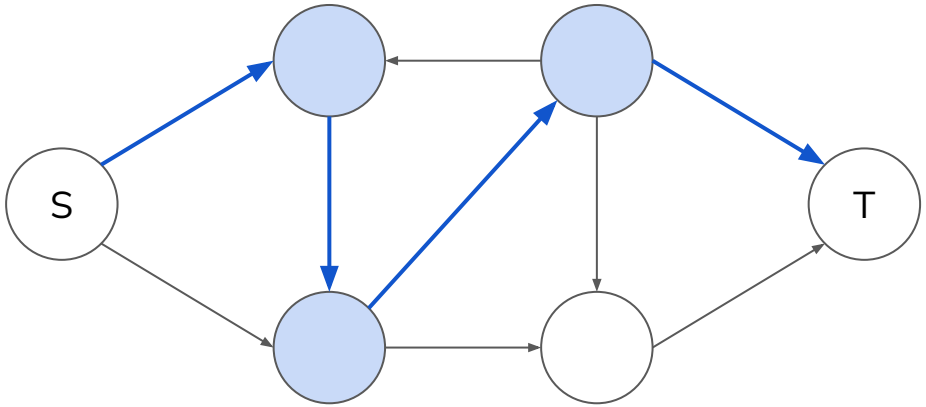
- Essentially the maximum flow from S to T!



Solving Graph Problems with Flow

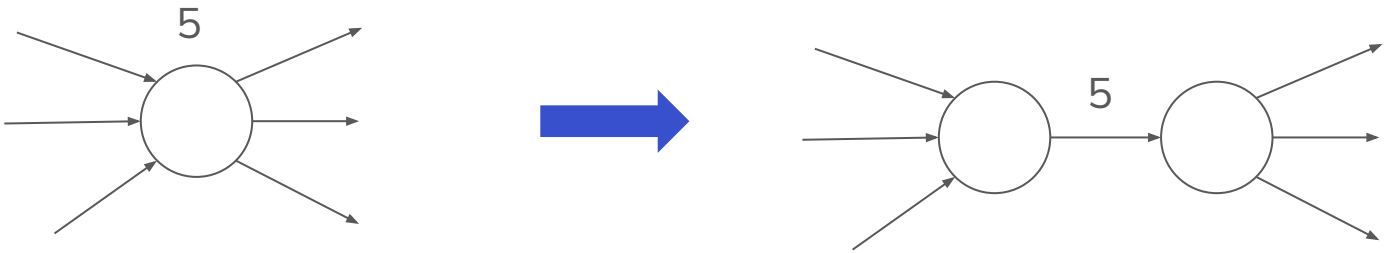
Vertex-disjoint paths: Find the maximum number of **vertex-disjoint paths** from the source to the sink, i.e. each **vertex (except S and T)** appears in at most one path.

- **Issue**: Flow only supports edge capacities but not vertex capacities.



Vertex Splitting Technique

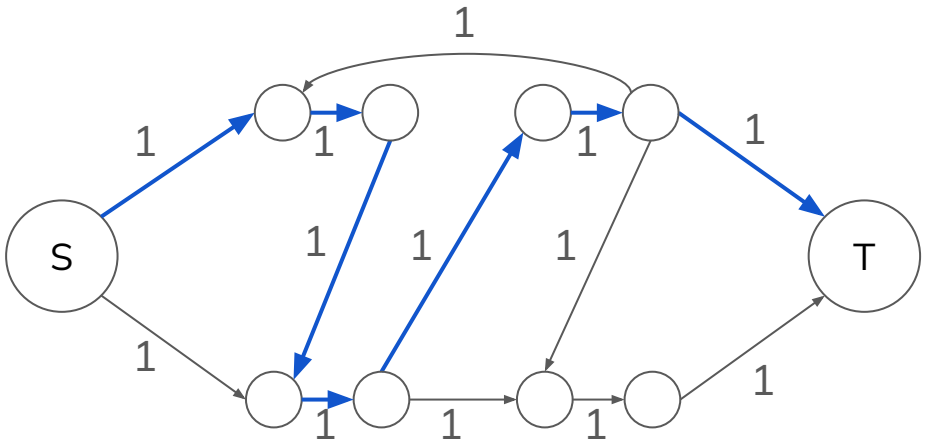
- Split a vertex into two vertices connected with an edge.
- By doing so, we can use the capacity of the edge to control the flow via the vertex!



Solving Graph Problems with Flow

Vertex-disjoint paths: Find the maximum number of **vertex-disjoint paths** from the source to the sink, i.e. each **vertex (except S and T)** appears in at most one path.

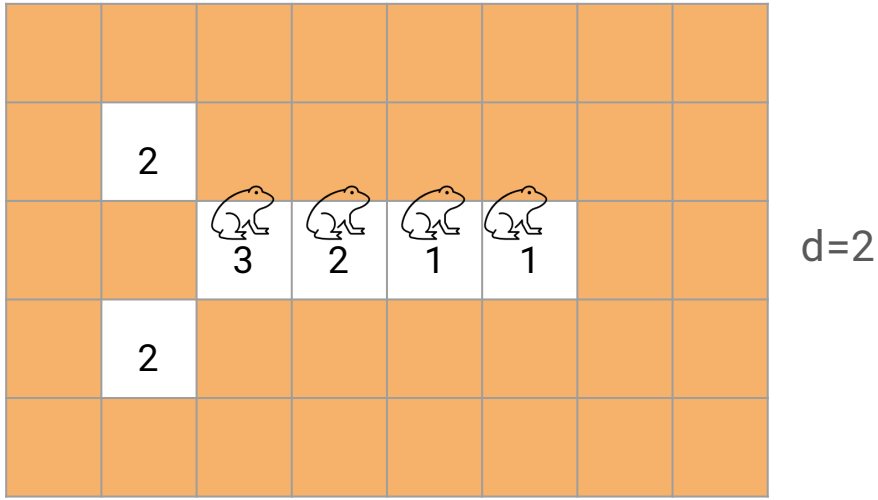
- Use the **vertex splitting technique** to support vertex capacities.



Problem Source: Luogu P2472

Modeling Non-Graph Problems with Flow

Some frogs are currently in a 2D grid. A frog can jump from its current location to another cell **no more than d units away** (in Euclidean distance). However, leaving a cell causes the cell's height to decrease by 1. A frog cannot jump to a cell with height 0. What is the maximum number of frogs that can jump outside the grid boundaries?



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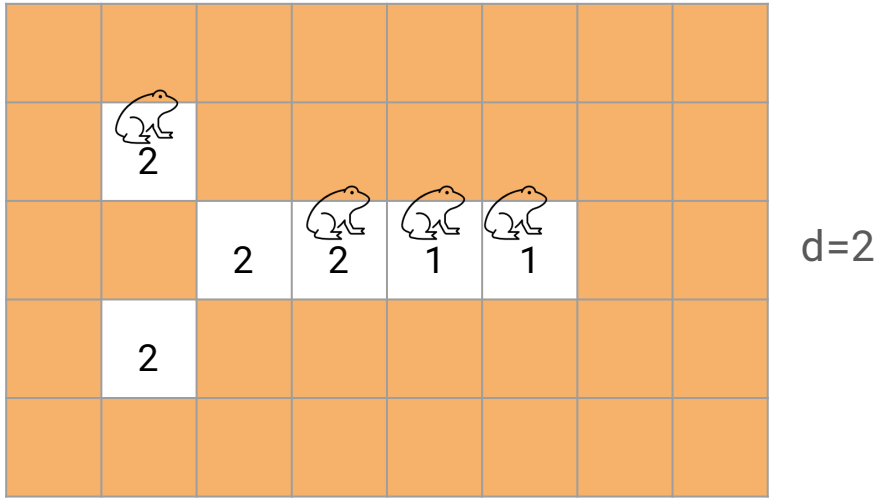

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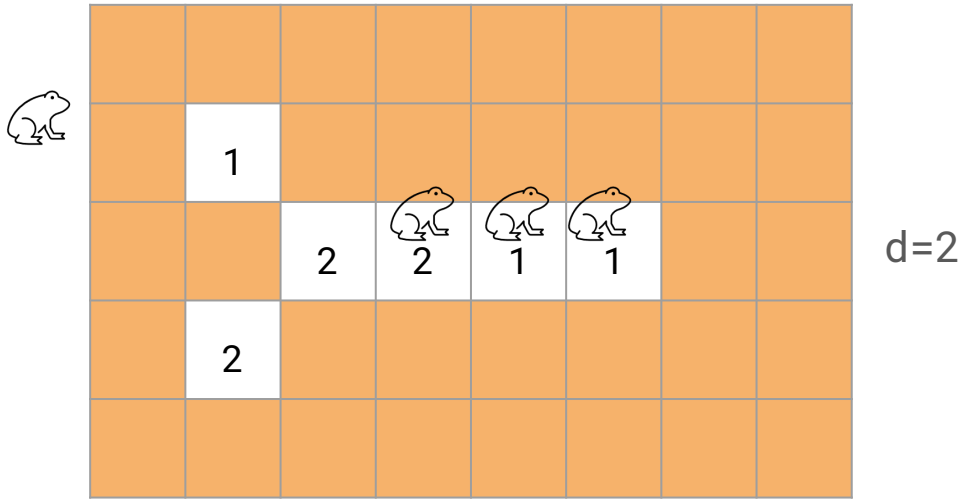

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Modeling Non-Graph Problems with Flow

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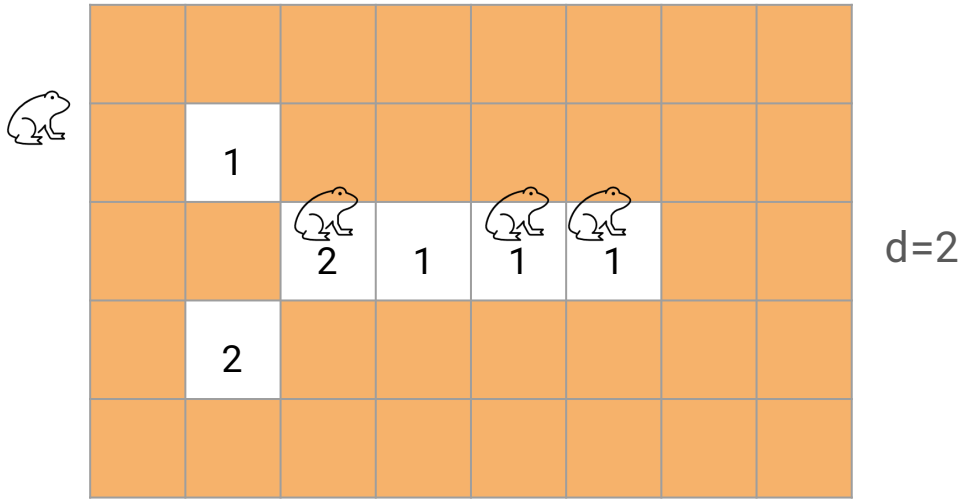

Problem Source: Luogu P2472

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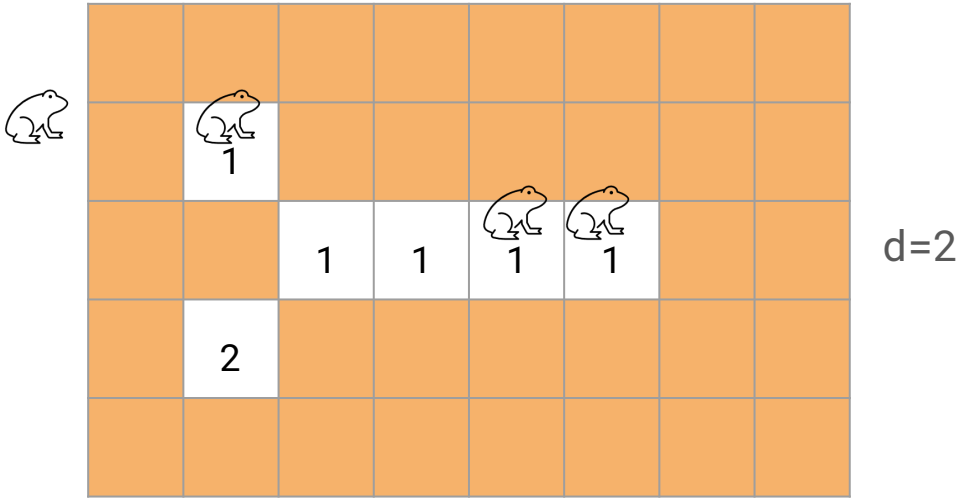

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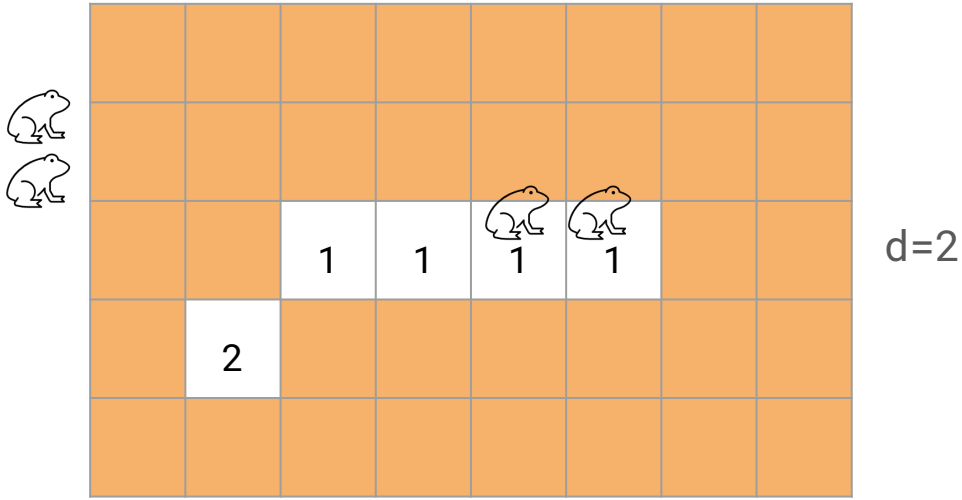

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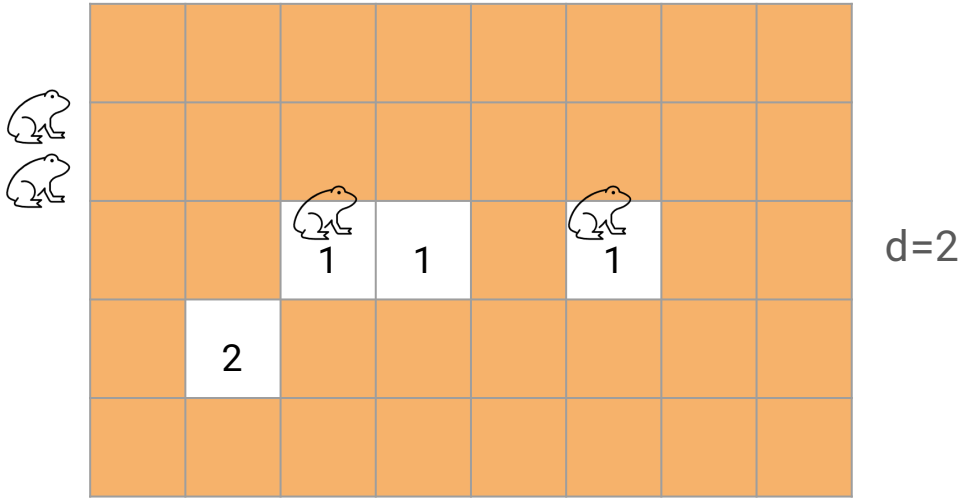

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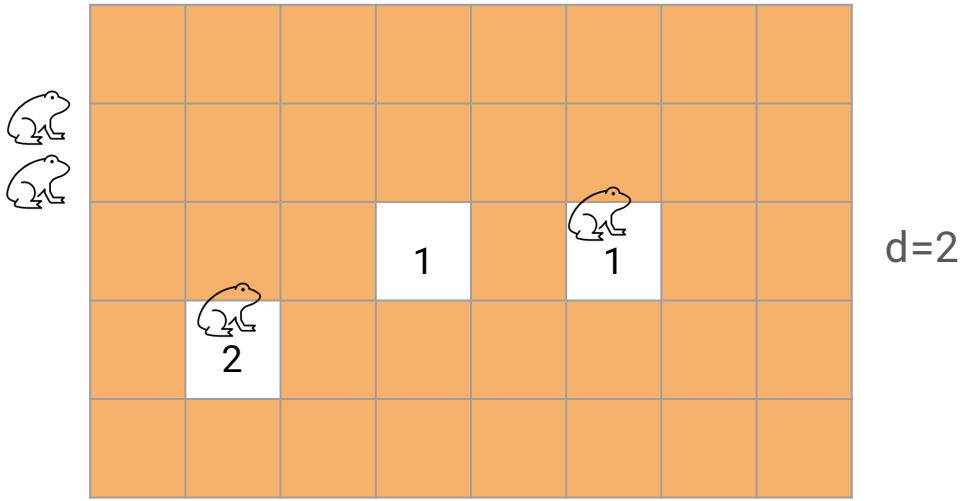

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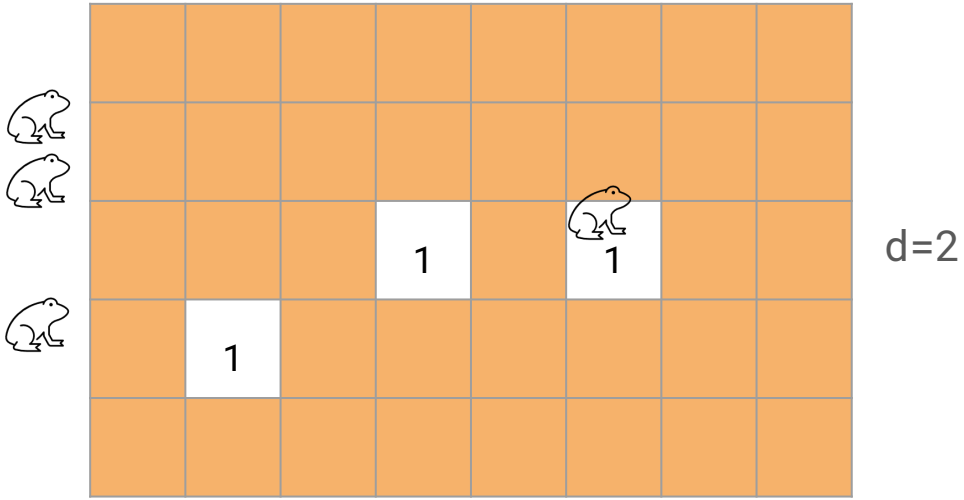

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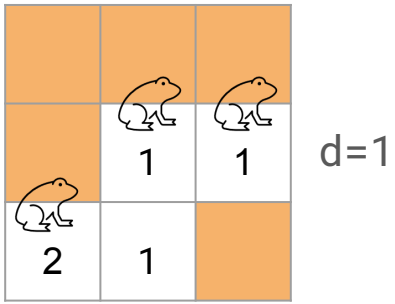
Modeling Non-Graph Problems with Flow

Let's try to model this into a flow problem.

- What does the vertices represent? Grid cells.
- What does the flow represent? Movement of the frogs.



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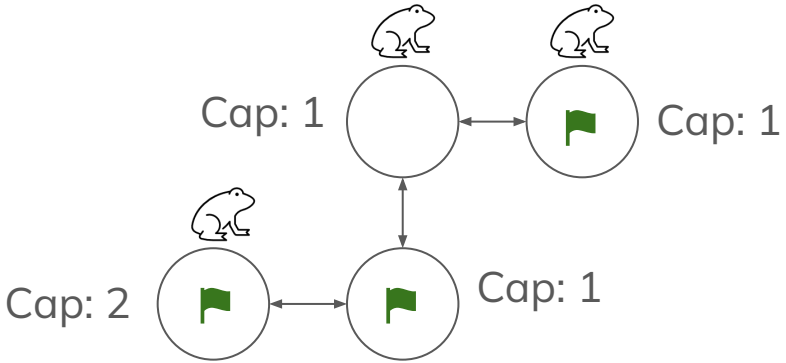
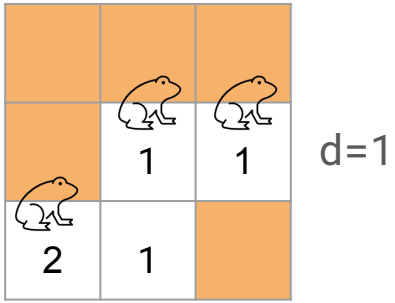


Problem Source: Luogu P2472

Modeling Non-Graph Problems with Flow

Issue 1: There are multiple starting and ending locations.



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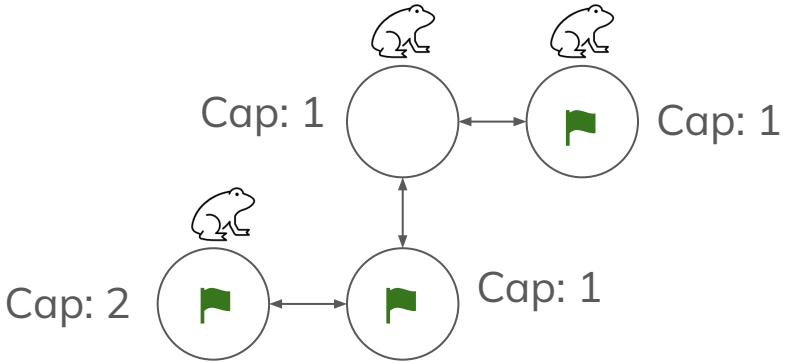
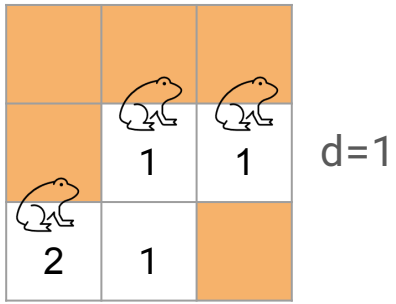


Problem Source: Luogu P2472

Modeling Non-Graph Problems with Flow

Issue 1: There are multiple starting and ending locations.

- Build a “super source” and a “super sink” connecting those.
- Capacity from source = 1 (the number of frogs). Capacity to sink = ∞.



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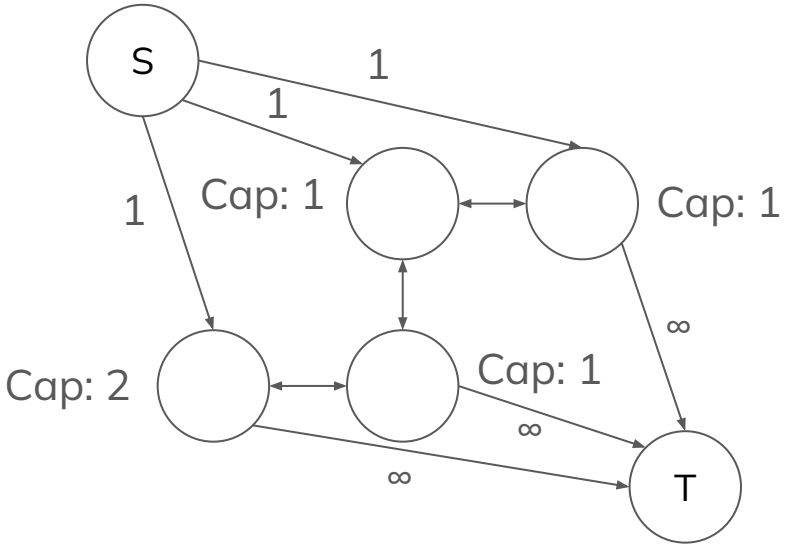

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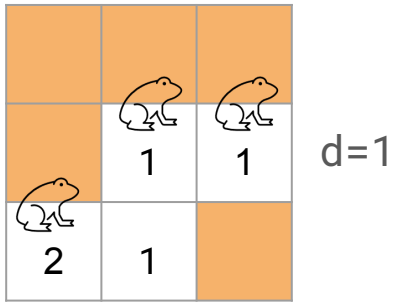
Modeling Non-Graph Problems with Flow

Issue 2: There are limits on the number of times each cell is used.



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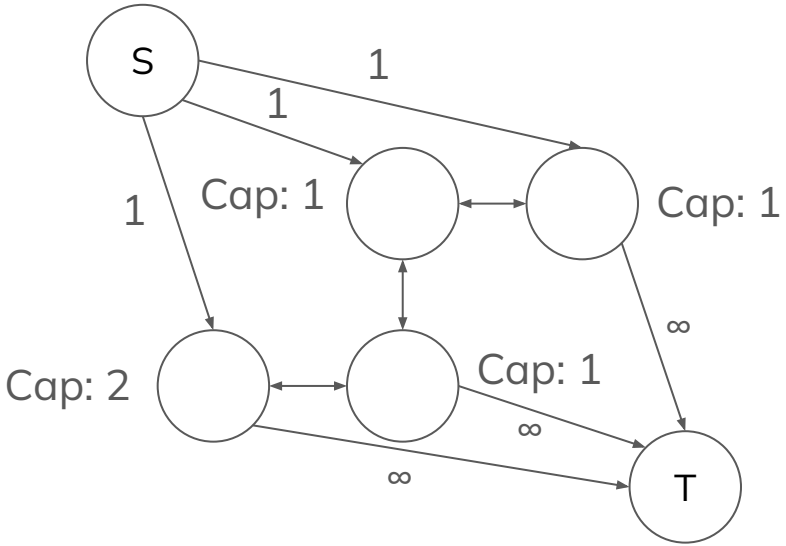

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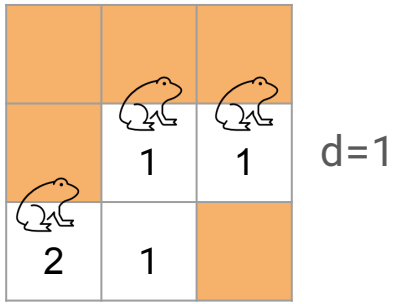
Modeling Non-Graph Problems with Flow

Issue 2: There are limits on the number of times each cell is used.

- Note that these are **vertex capacities.** Use the **vertex splitting technique**.



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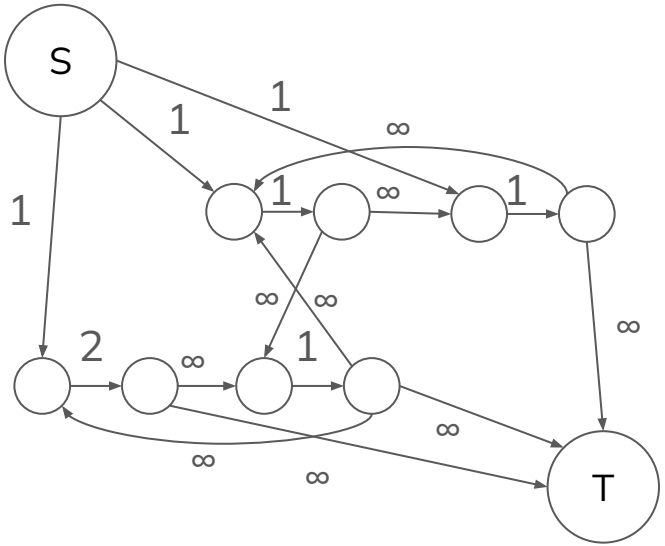
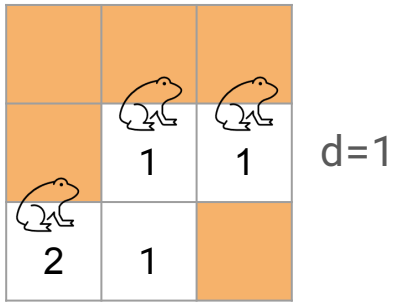


Problem Source: Luogu P2472

Modeling Non-Graph Problems with Flow

Solution:

- Build the flow network for the given input grid.
- Answer = **Maximum flow** from S to T. // O(EF) = O(Grid Size^2 × Number of frogs)



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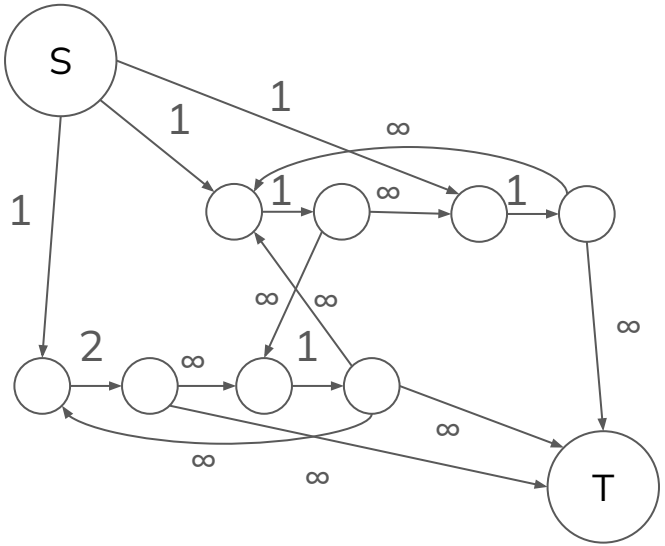


Summary

To solve **maximum flow** problems:

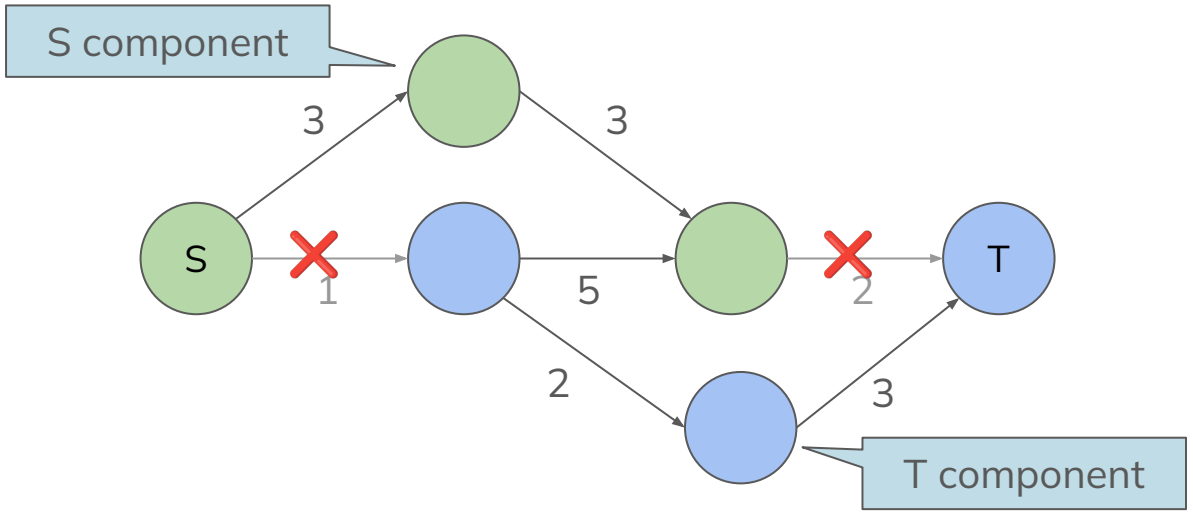
- Find out what the **vertices** and the **flow** (→ edges) should represent.
- Connect all “starting locations” and “ending locations” (goals) with a **super source** and a **super sink**.
- Convert all vertex capacities into edge capacities with the **vertex splitting trick**.
- Run Ford–Fulkerson Algorithm to find the maximum flow.

- HKOJ Practice Tasks: A410, A411, A412

Minimum Cut

Minimum Cut Problem

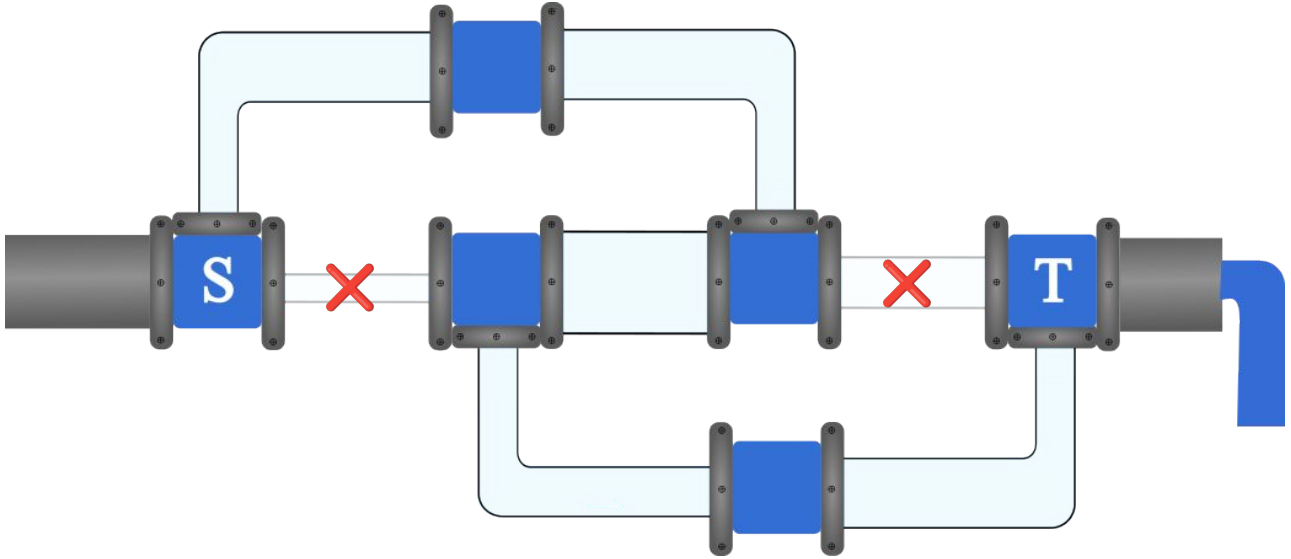
You are given a **weighted** graph. Delete some edges with **smallest total weight** to disconnect the sink from the source (i.e. there are no more paths from S to T).



Max-Flow Min-Cut Theorem

Let's rewrite it using **flow** terms.

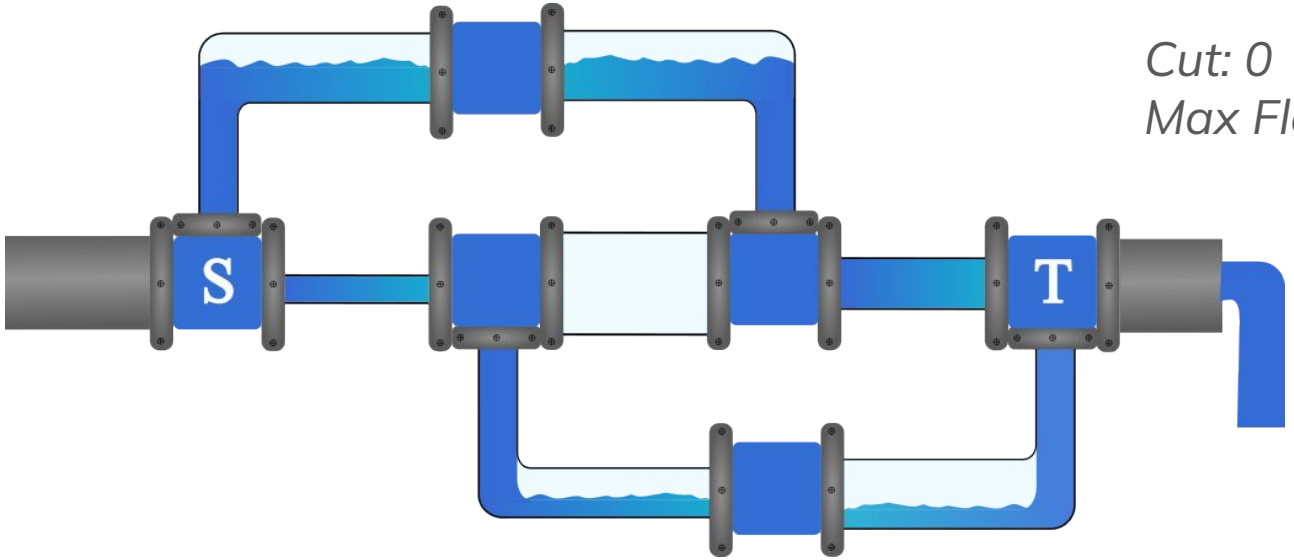
You are given a flow network. Delete some edges with **smallest total capacity** to disconnect the sink from the source (i.e. the maximum flow from S to T is 0).



Max-Flow Min-Cut Theorem

First Impression:

- By cutting an edge with capacity x, we can only reduce the flow by x at best.
- Therefore, **minimum cut ≥ maximum flow**. (They are related!)



Cut: 0
Max Flow: 3

Max-Flow Min-Cut Theorem

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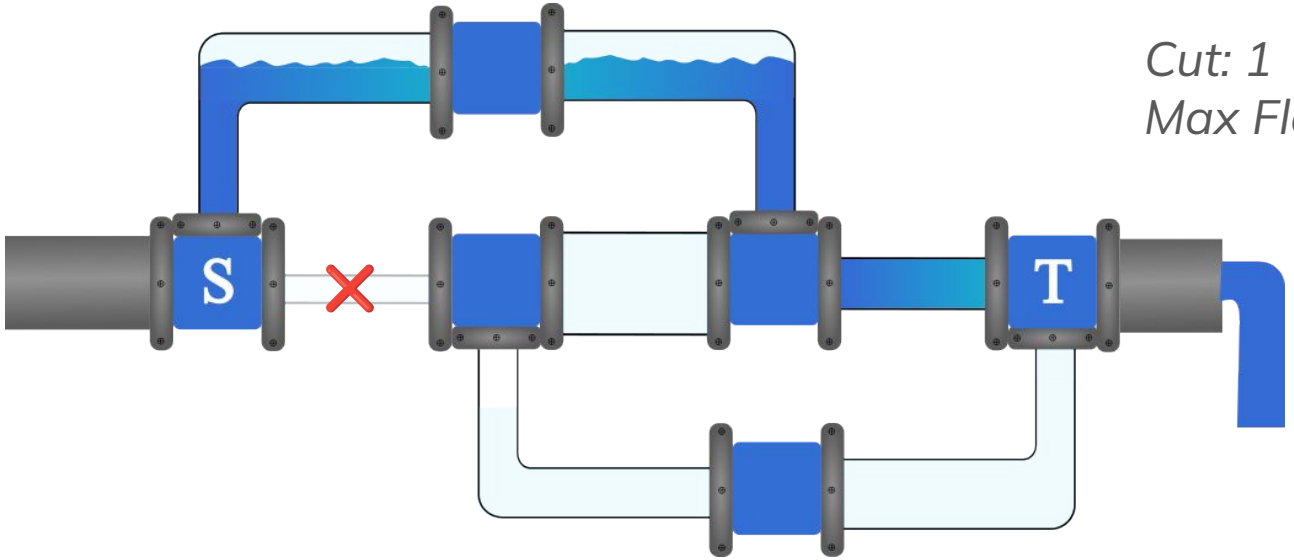

Cut: 1
Max Flow: 2

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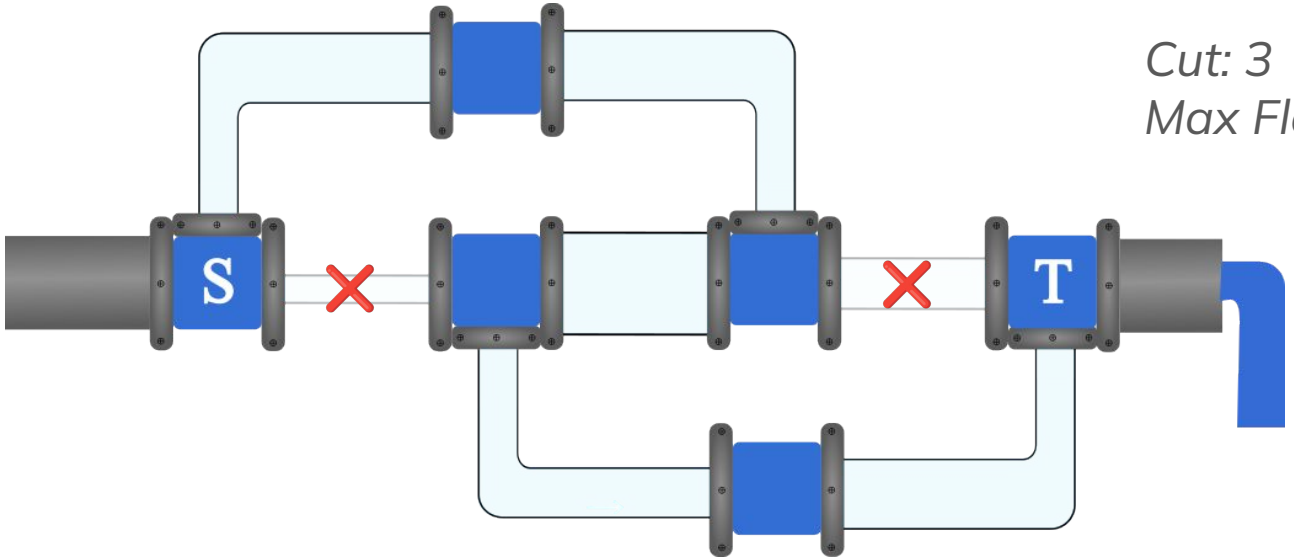
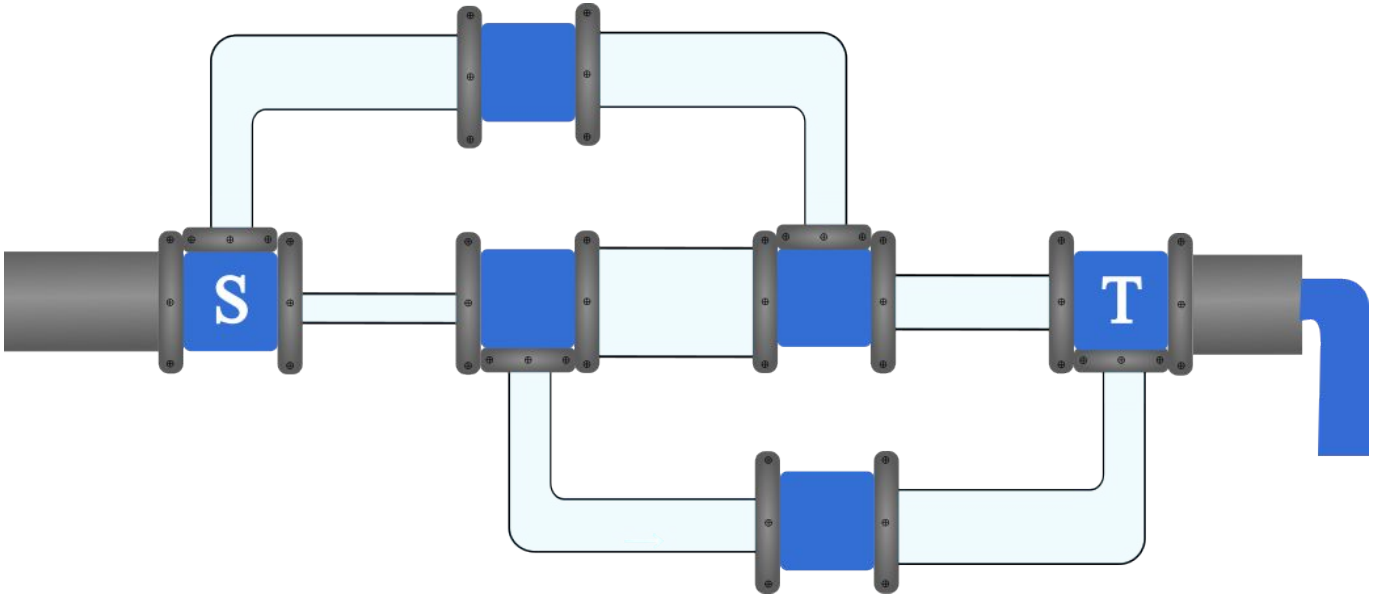


Cut: 3
Max Flow: 0

Max-Flow Min-Cut Theorem

Can we find the **minimum cut** from the **maximum flow**?



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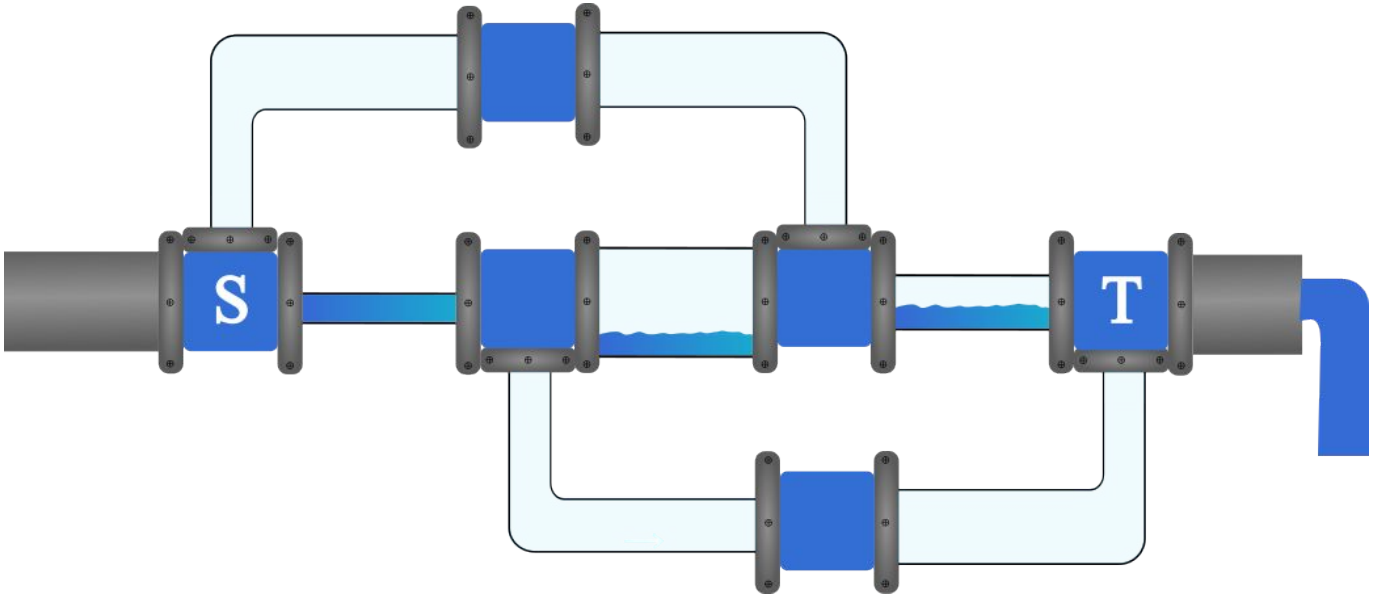

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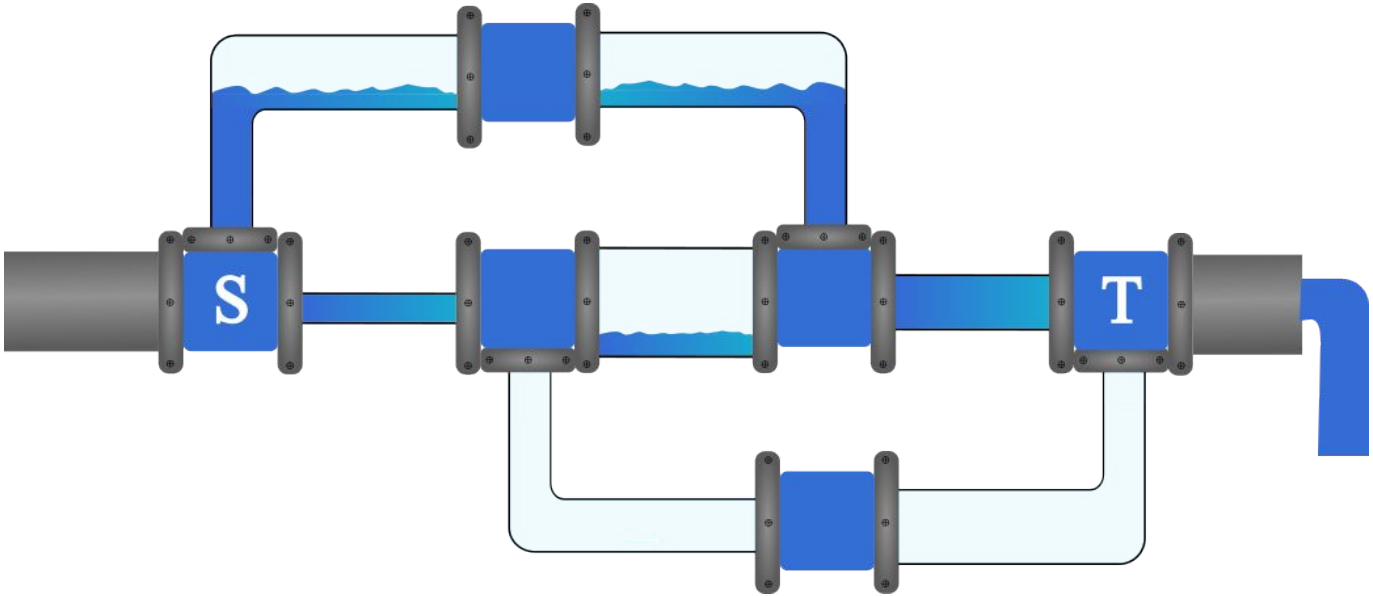

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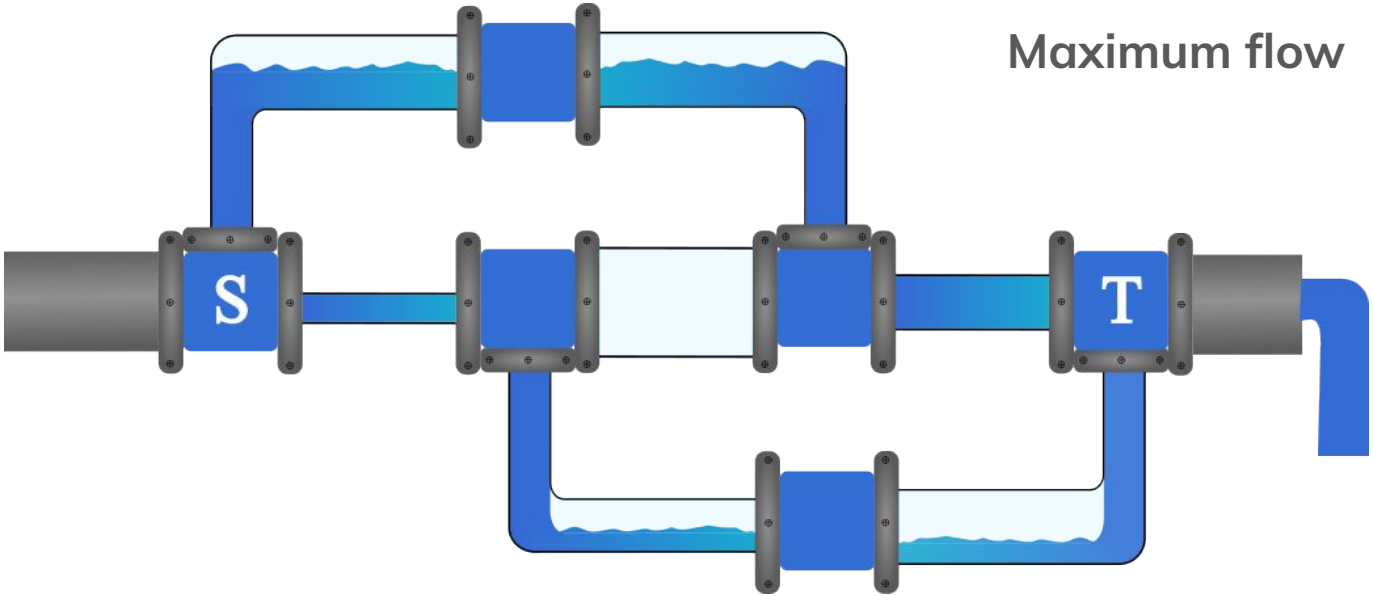

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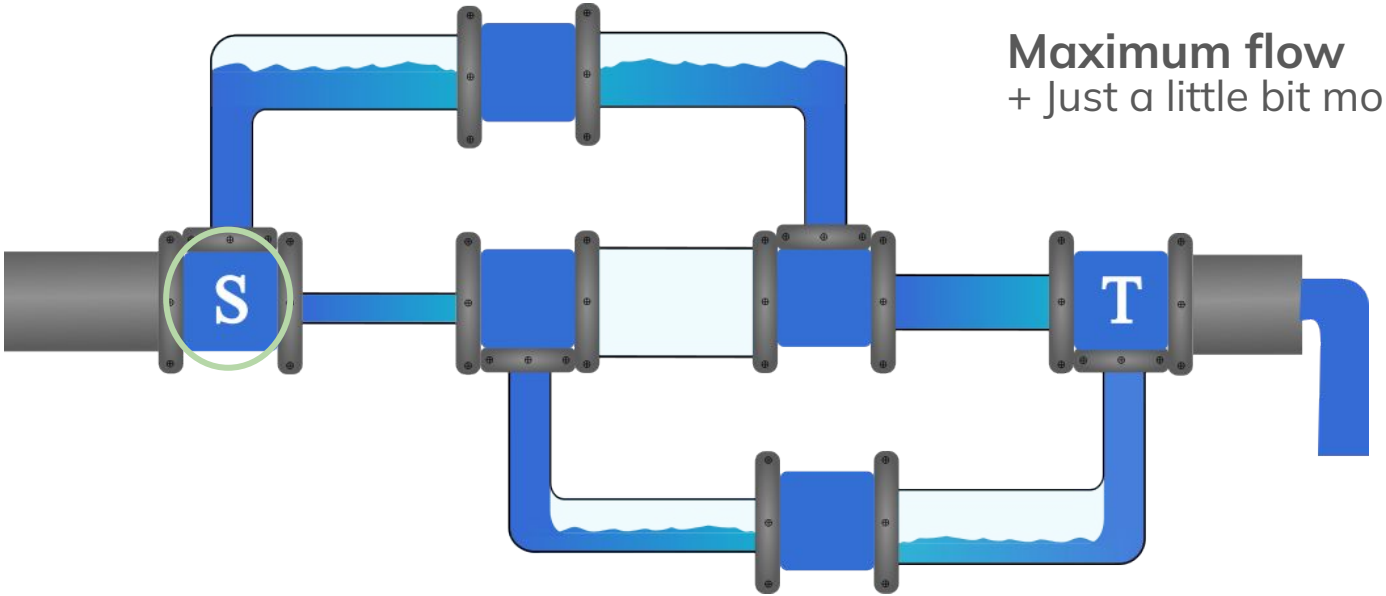

Maximum flow
+ Just a little bit more...

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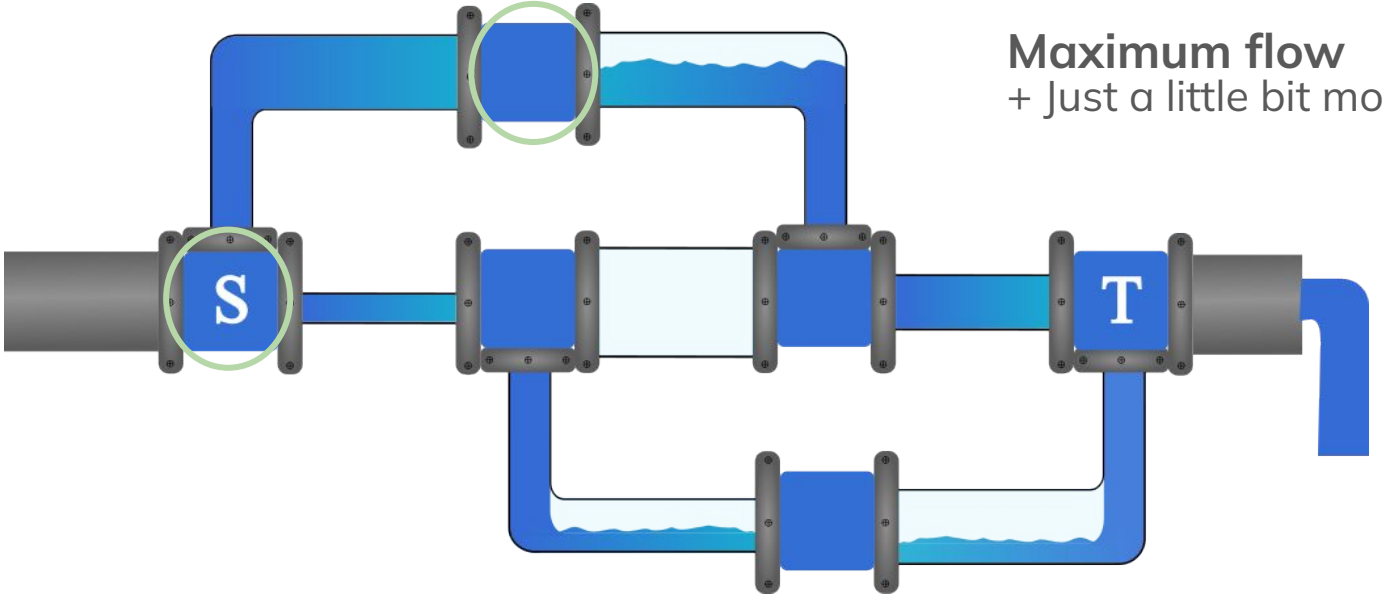

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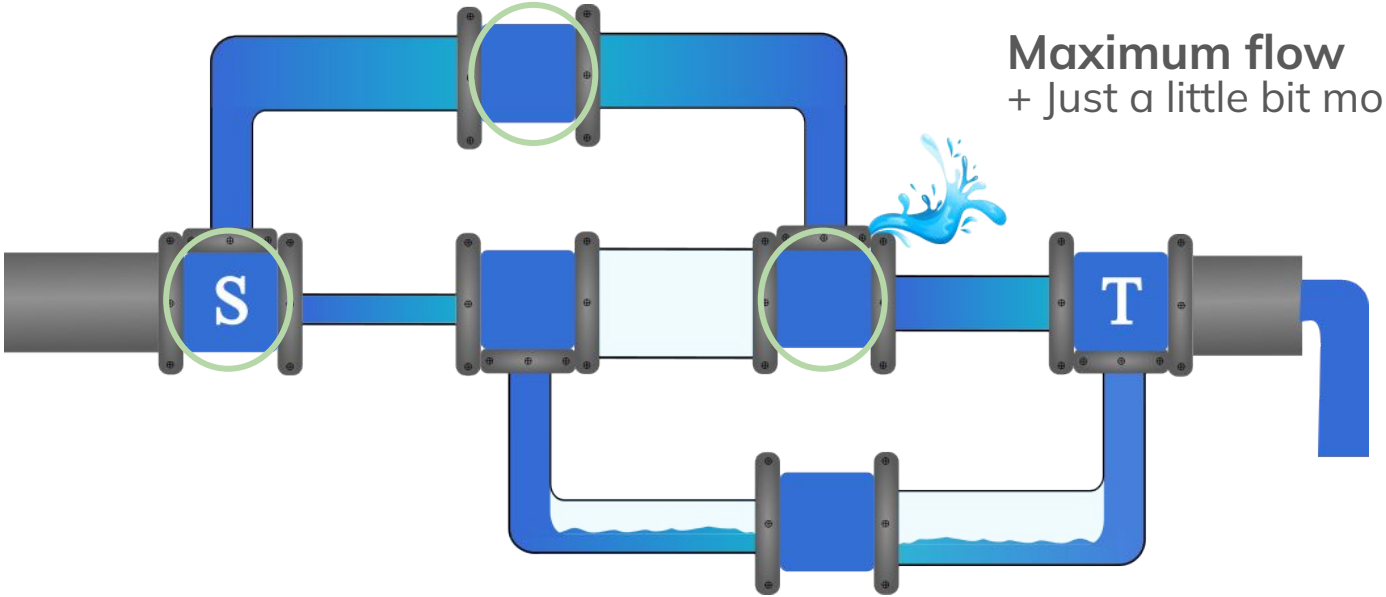

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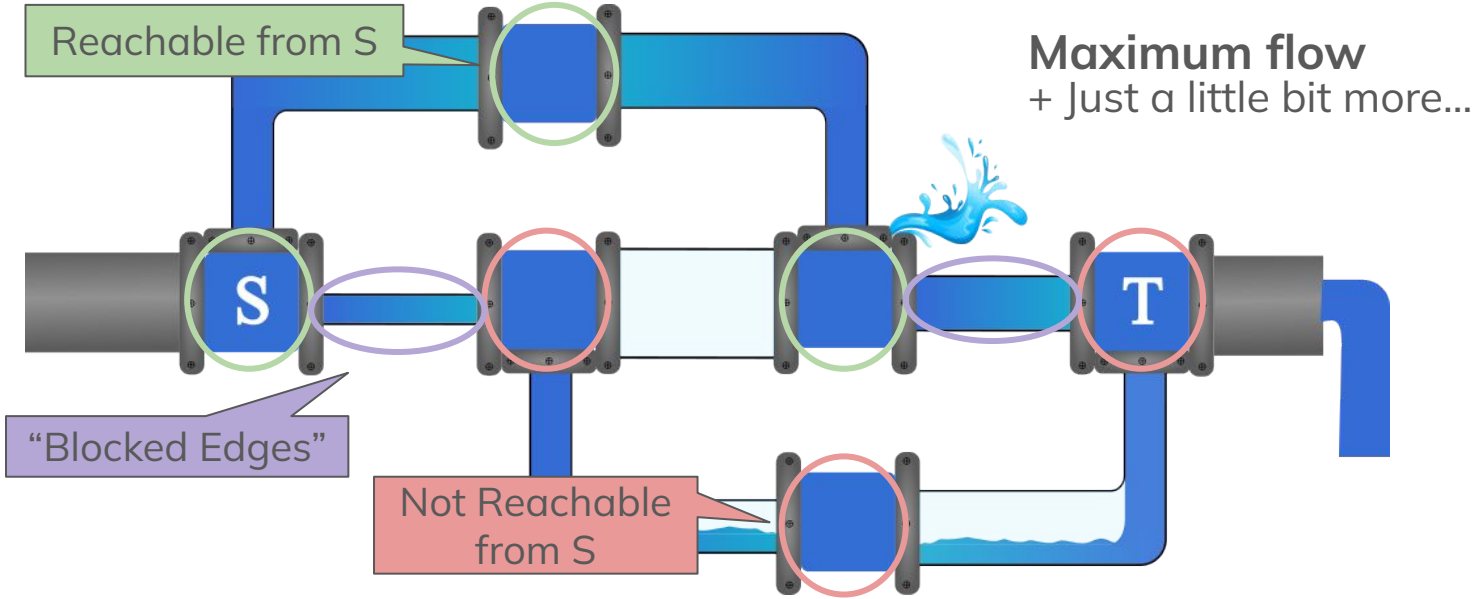

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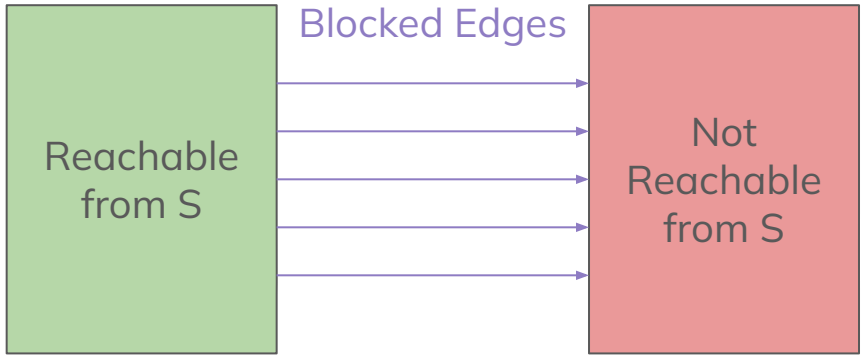

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(Otherwise, they do not “block”.)

Max-Flow Min-Cut Theorem

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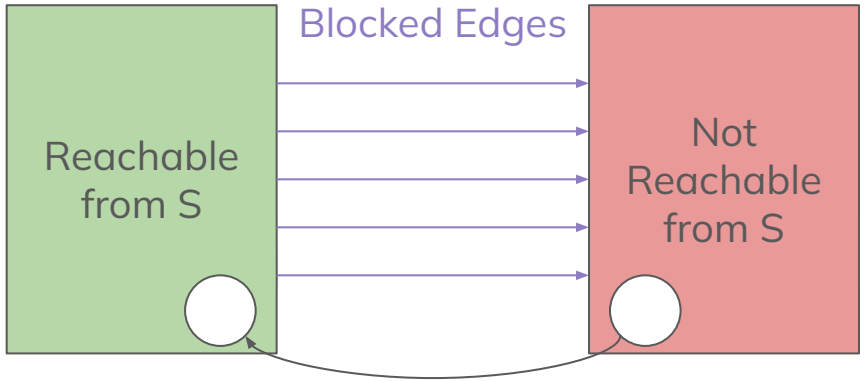

Can we have flow in this direction?

Observation 1: The flow in all blocked edges are at their maximum capacities.

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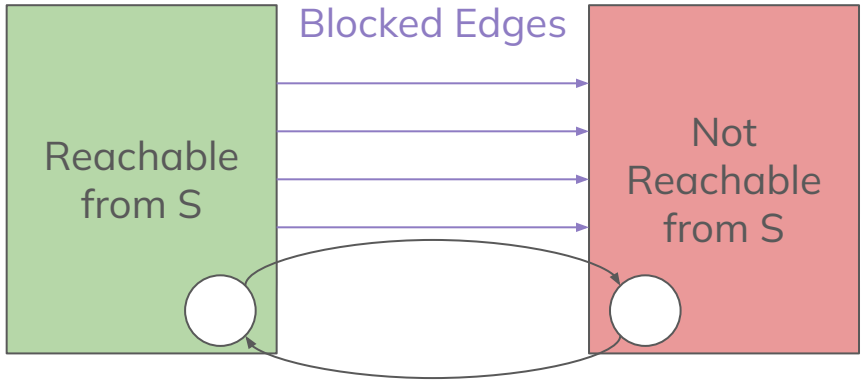
Observation 2: If there are flow from red to green area, we can "undo" it to "expand" the green area accordingly.

Max-Flow Min-Cut Theorem

Can we find the **minimum cut** from the **maximum flow**?



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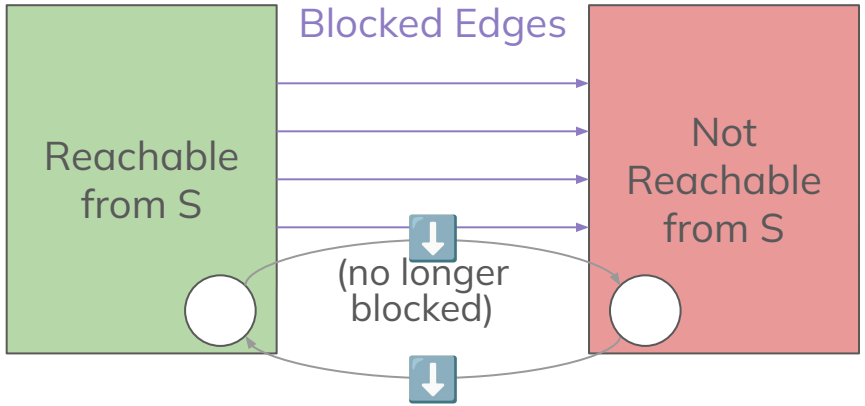

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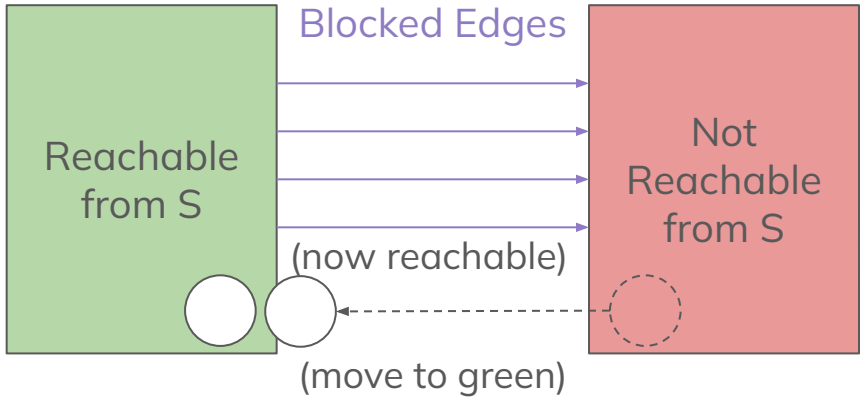

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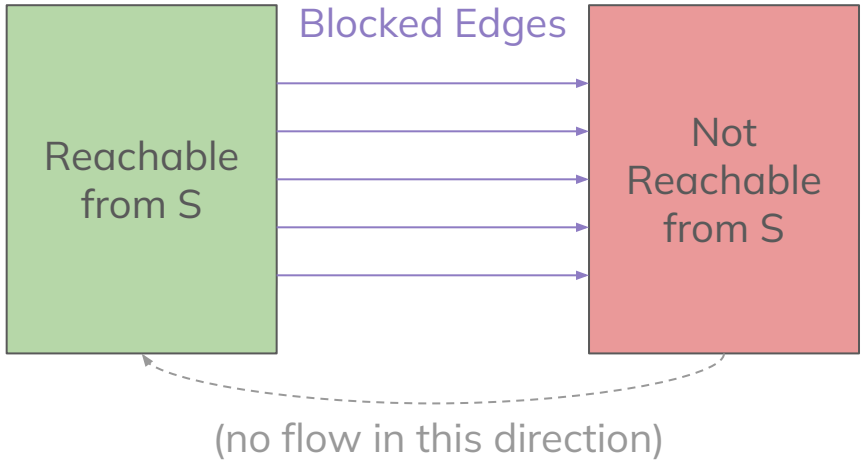
(...we can repeat this process until no more flow from red to green)

Max-Flow Min-Cut Theorem

Can we find the **minimum cut** from the **maximum flow**?



Observation 1: The flow in all blocked edges are at their maximum capacities.

(Otherwise, they do not "block".)

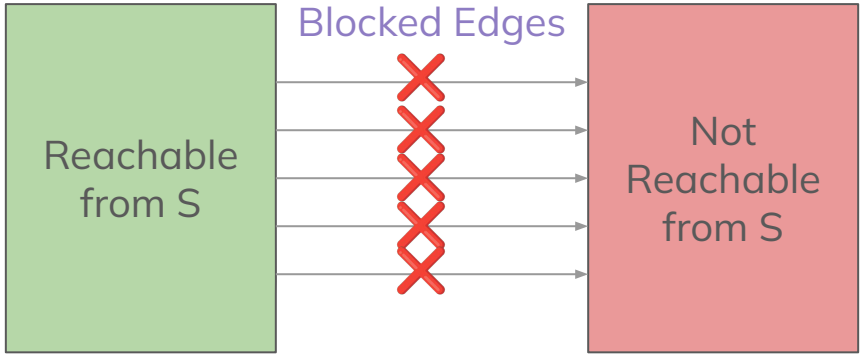
Observation 2: If there are flow from red to green area, we can "undo" it to "expand" the green area accordingly.

Result: The sum of weights of all blocked edges = The maximum flow.

(By flow conservation, flow does not magically "appear" or "disappear" in the red area.)

Max-Flow Min-Cut Theorem

- By cutting all blocked edges, we can attain a cut equal to the maximum flow.
- Is this optimal?
  Yes, since minimum cut ≥ maximum flow.



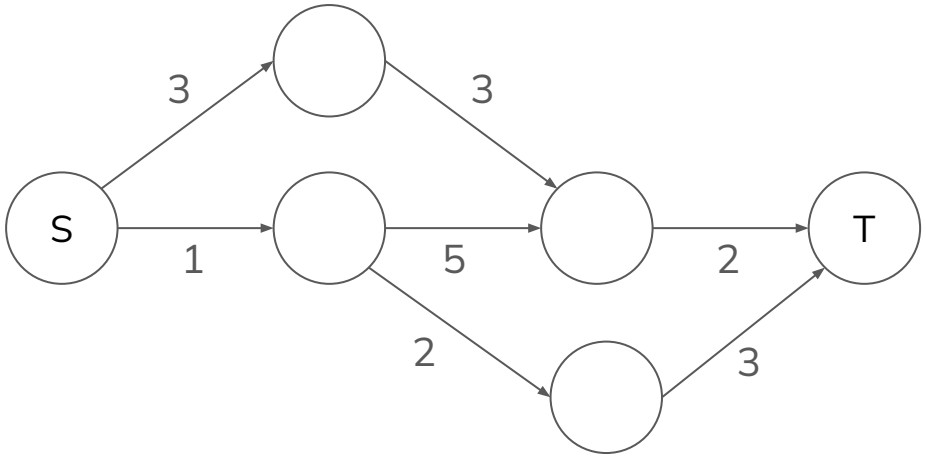
Max-Flow Min-Cut Theorem

The maximum flow from S to T is **equal to** the minimum cut from S to T.

Minimum Cut – Implementation

1. Find the maximum flow (Minimum cut = Maximum flow)
2. Find all vertices reachable / not reachable from S in the **residual network**
3. The cut consists of all edges connecting from a vertex reachable from S to a vertex not reachable from S

[Original Network]



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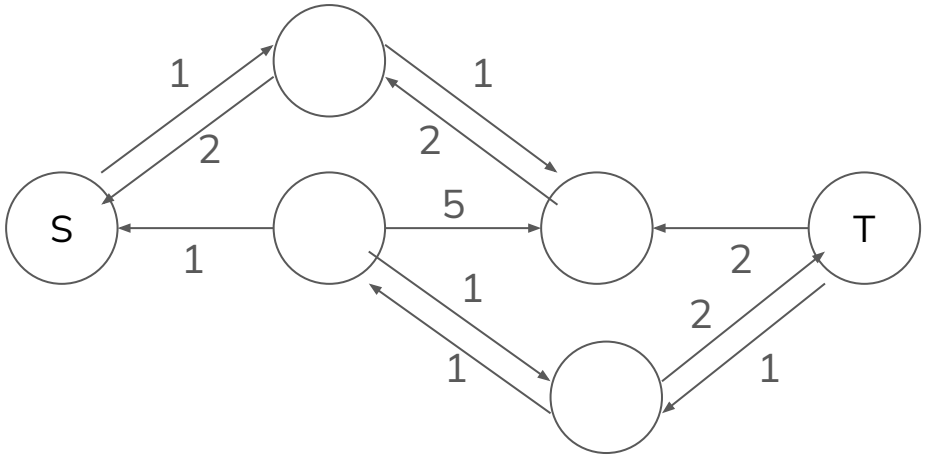
[Residual Network]

Max Flow = 3



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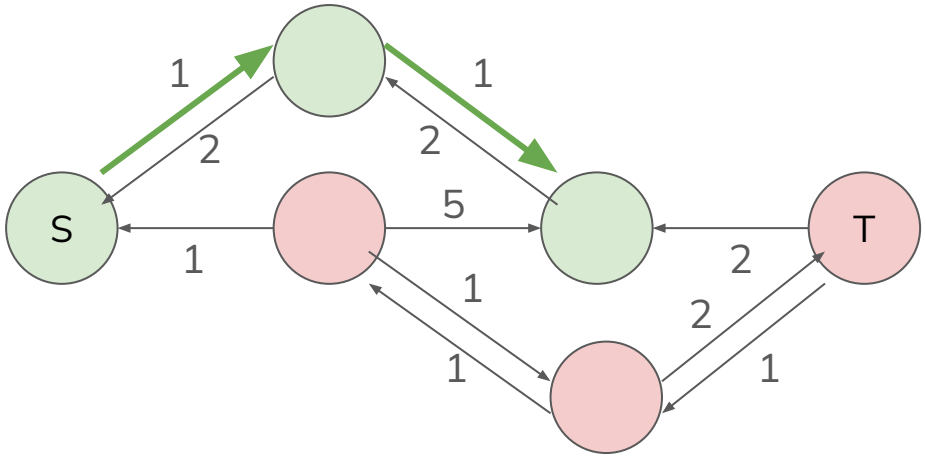

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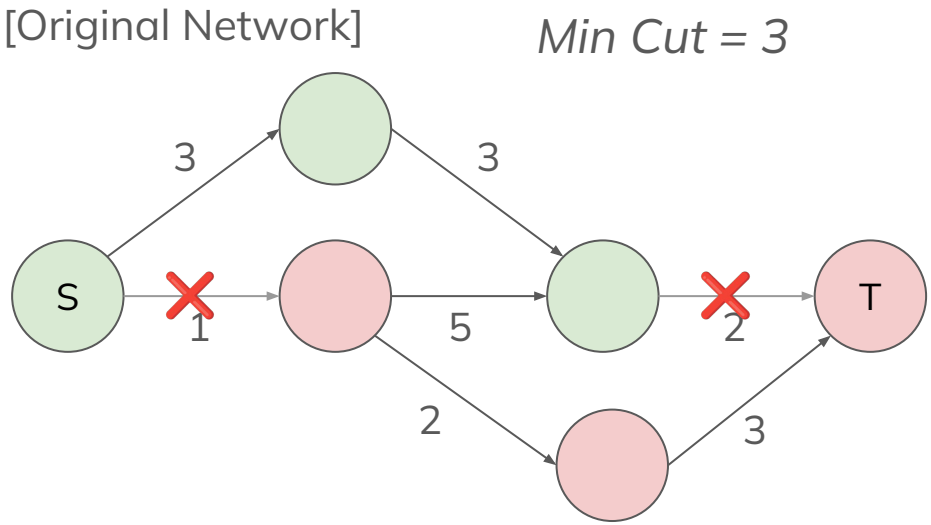
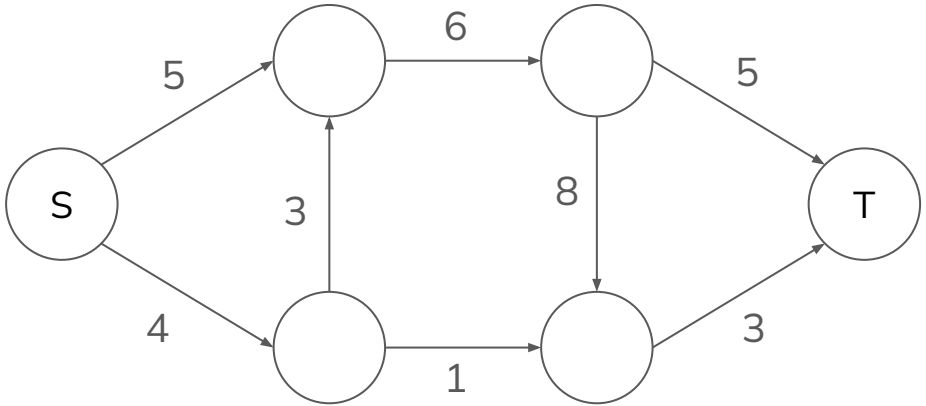


Quick Check

🔤 Consider the same flow network. Which edges are in the minimum cut from S to T?



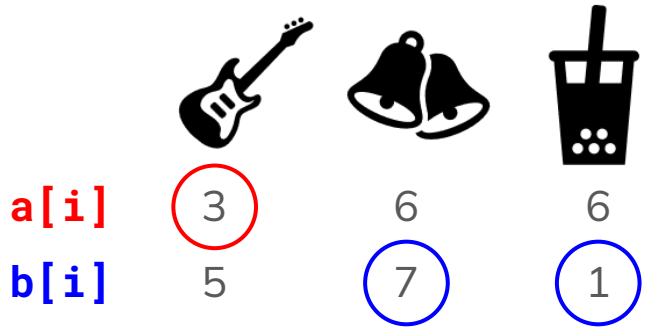
(Answer: 1 and 6)

Set Partitioning Model

There are N items. You should place each item into either the **red** bag or the **blue** bag.

- Putting item `i` in the red bag costs `a[i]`.
- Putting item `i` in the blue bag costs `b[i]`.
- There are `M` constraint pairs `(u, v, w)`. If item `u` is red and item `v` is blue, there is an additional cost of `w`.

What is the minimum possible cost to place all N items?



Constraint Pairs:

1.  `w = 1`
2.  `w = 10`

Set Partitioning Model

To model this as a minimum cut problem, we need to think in a special way:

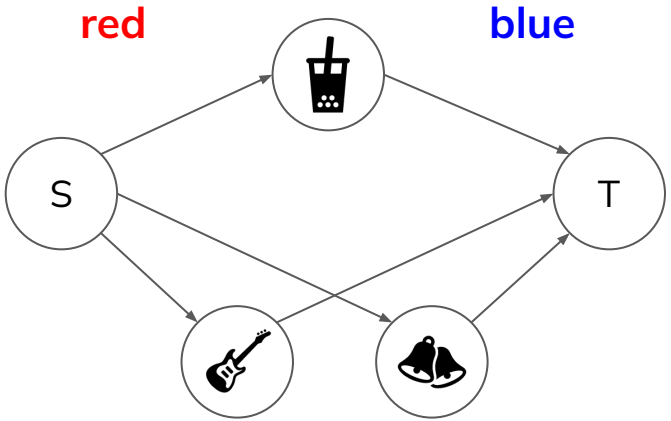
"There will be a path from S to T **if something goes wrong**."

- **Cutting an edge** is same as **paying some cost**.
- Our goal is to disconnect T from S → nothing goes wrong.

Set Partitioning Model

Our Requirements:

1. Each item must be placed in either the red bag or the blue bag.
   Create a node for each item. Connect it to both S and T.
   Cutting an edge → placing the item into a bag.



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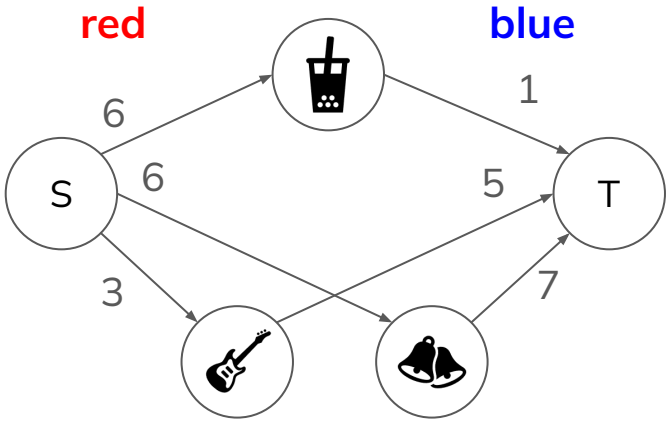
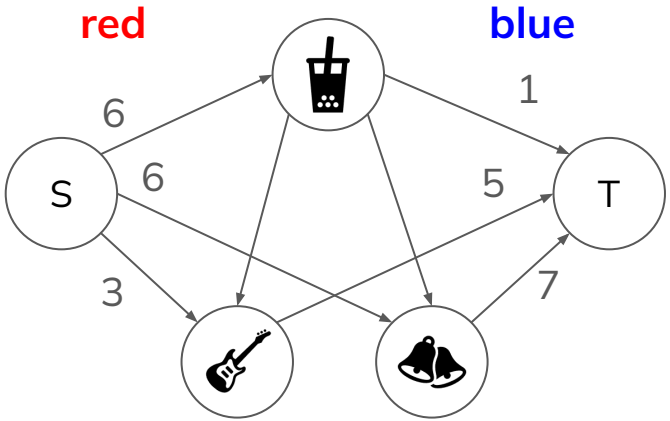


Cost of cutting the edge
= Cost of placing the item

Set Partitioning Model

Our Requirements:

2. There are no conflicts **OR** we paid for the conflicts.
   If item u is **red** and item v is **blue**, then we have the edges S → u and v → T **cut** and the edges u → T and S → v **remaining**. Build an edge v → u to connect from S to T.



Set Partitioning Model

Our Requirements:

2. There are no conflicts **OR** we paid for the conflicts.
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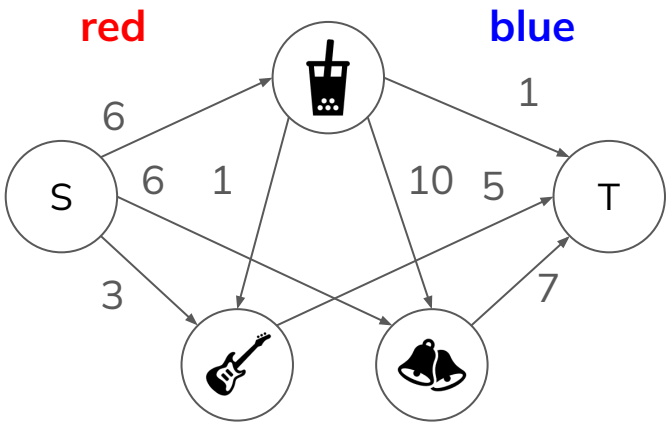
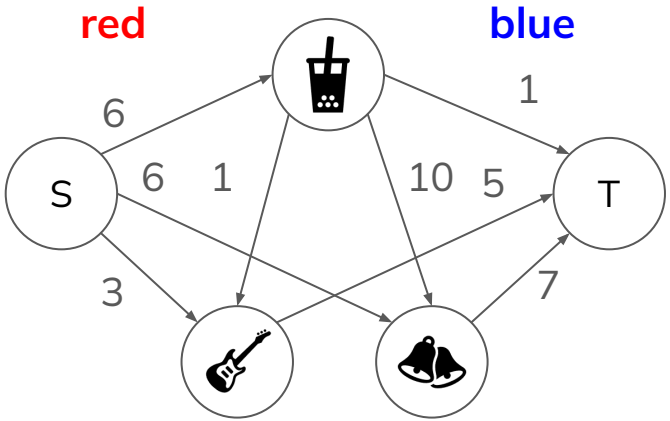


Cost of cutting the edge
= Cost of the conflict

Set Partitioning Model

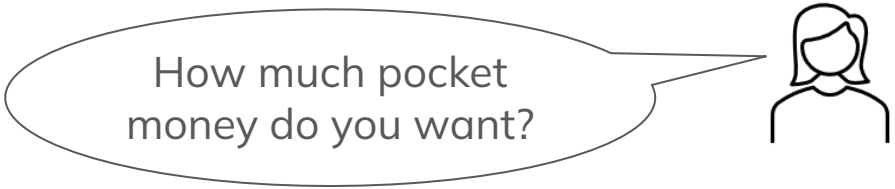
Solution:

- For each item, build an edge S → `i` with weight `a[i]` and `i` → T with weight `b[i]`.
- For each constraint, build an edge v → u with weight `w`.
- Answer = Minimum Cut = Maximum Flow.



“Optimistic Thinking”

Imagine you are negotiating for pocket money with your parents.



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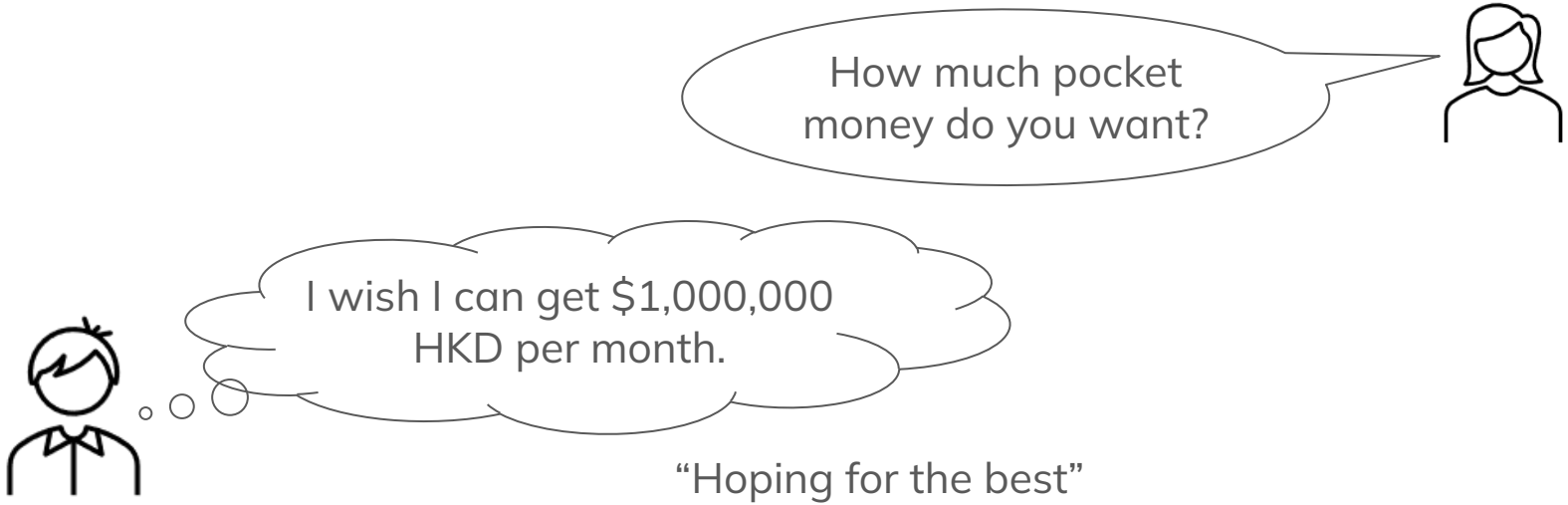

"Hoping for the best"

"Optimistic Thinking"

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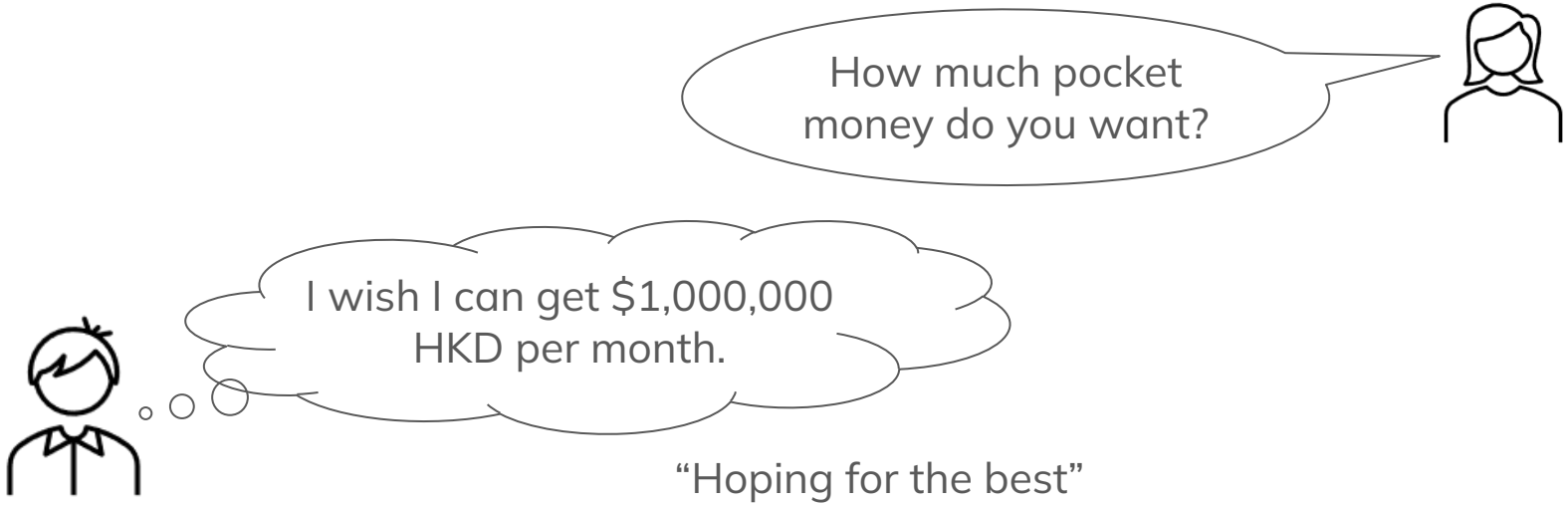

"Hoping for the best"

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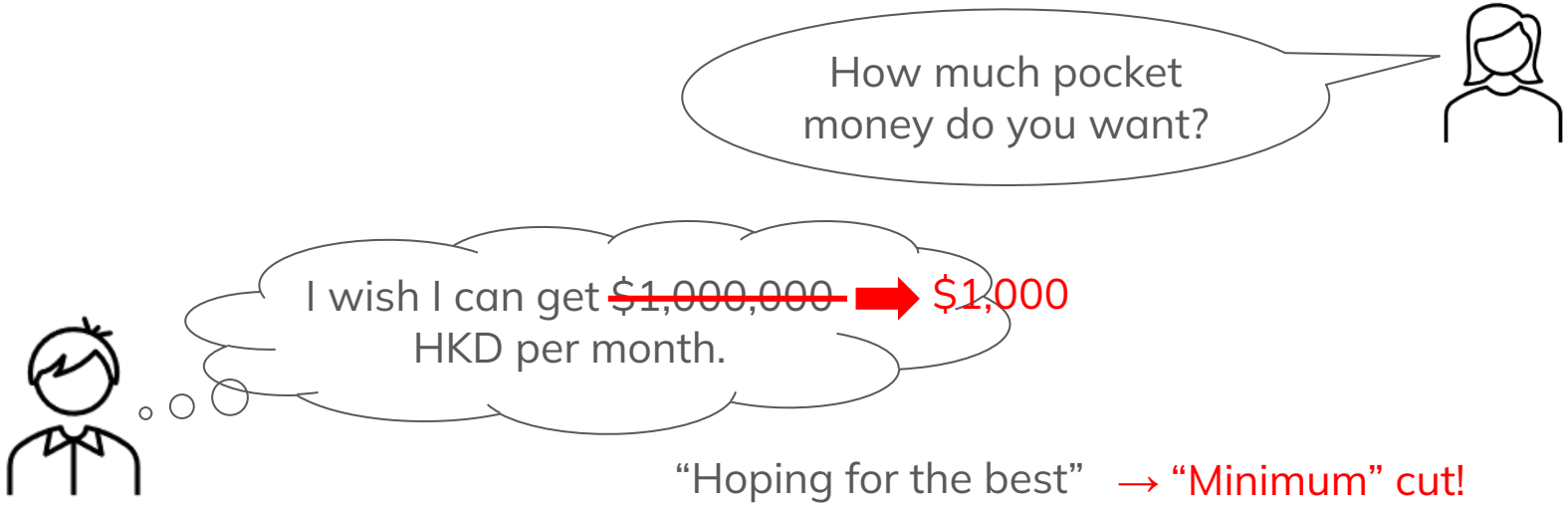


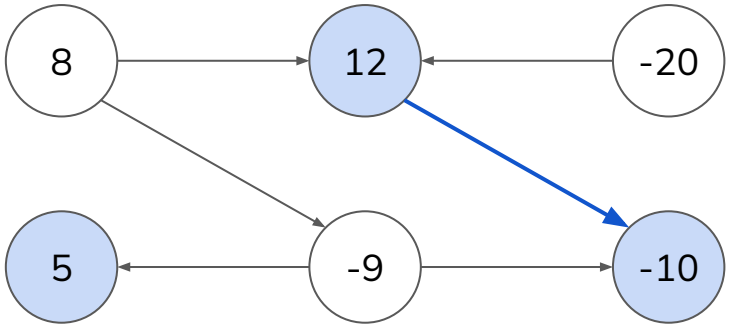
"Hoping for the best" → "Minimum" cut!

Modeling Problems with Cut

Maximum Weight Closure: Given a **directed** graph where each **vertex** has a **weight**, pick some vertices subject to the following condition for each edge u → v:

"If you picked vertex u, you must pick vertex v as well."

Maximize the sum of weights of the vertices picked (the weights can be negative).

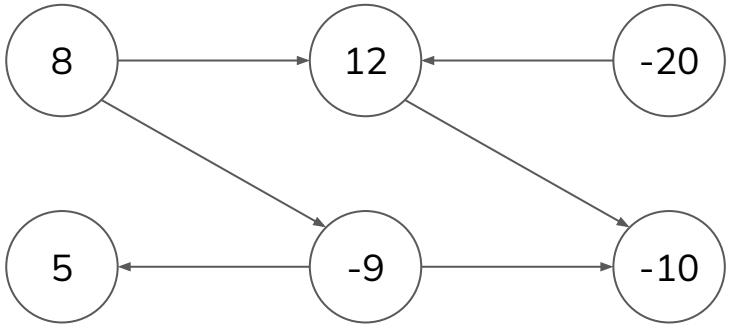


Sum = 12 + 5 - 10 = **7**

Modeling Problems with Cut

Let's apply "optimistic thinking":

What is the ***best ever scenario*** you can wish for?



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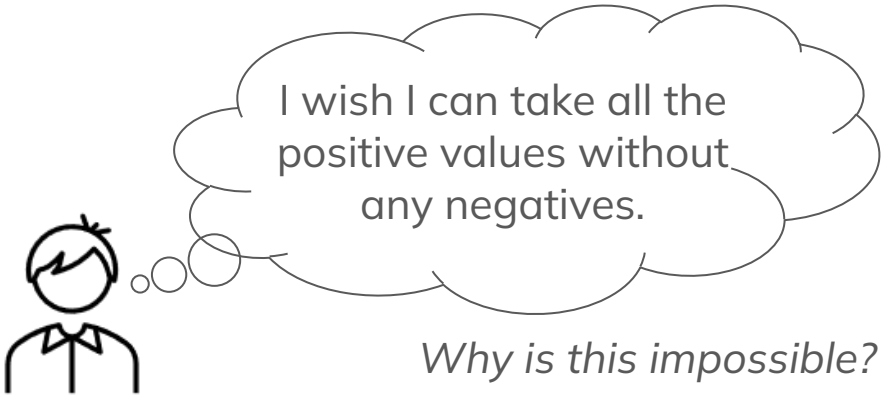
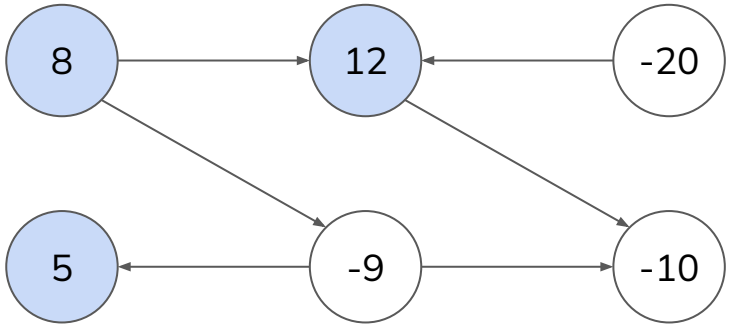


Why is this impossible?



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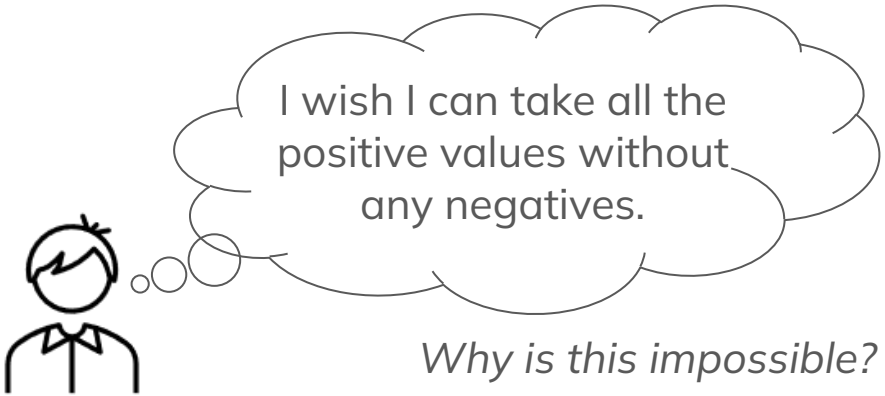

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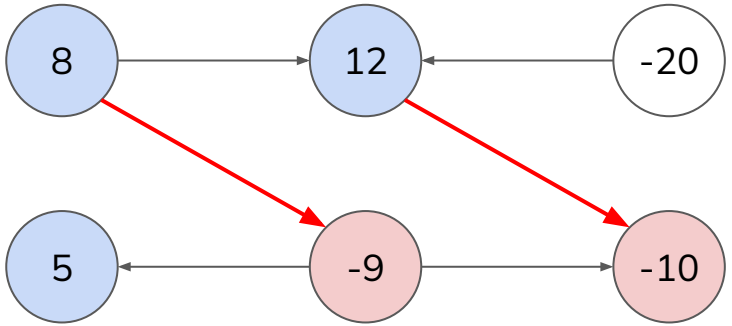
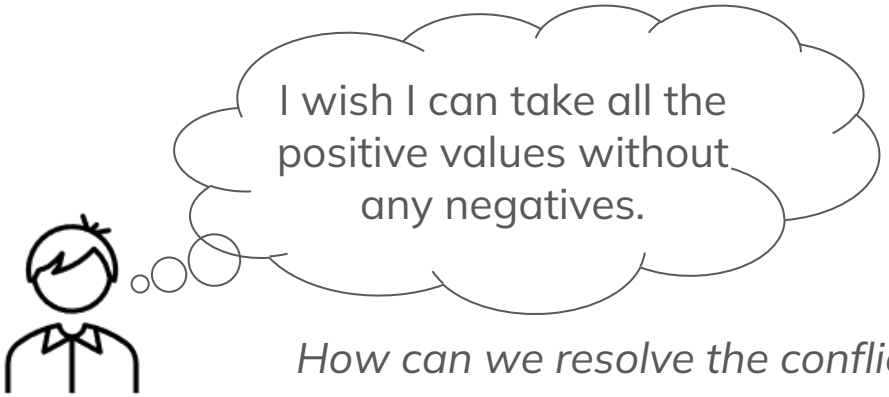


Conflicts: We picked u without picking v.
→ Express this as a **S-T path**.

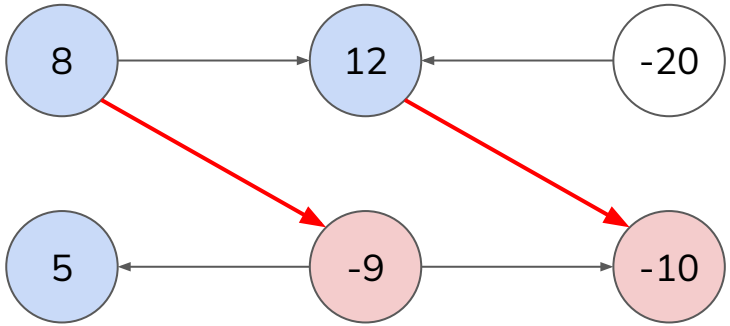
Modeling Problems with Cut

Let's apply "optimistic thinking":

What is the ***best ever scenario*** you can wish for?



How can we resolve the conflict?



Modeling Problems with Cut

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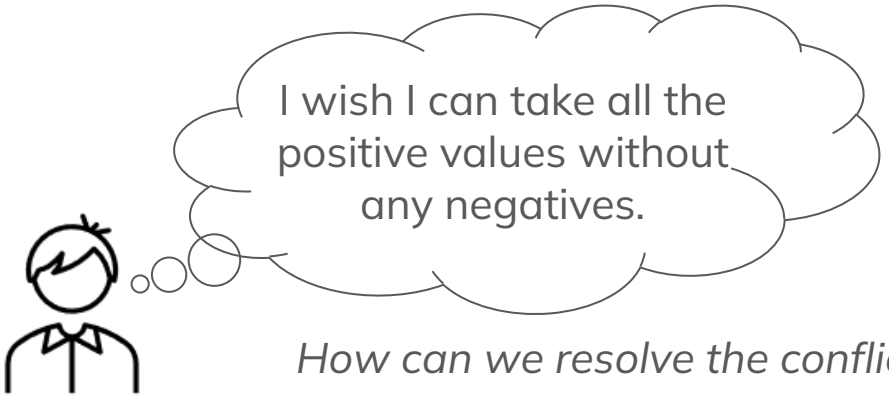

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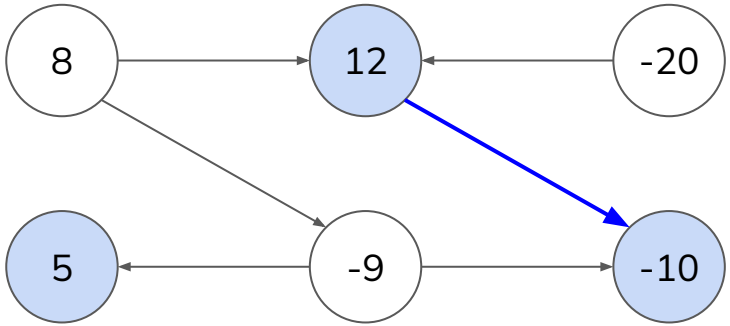
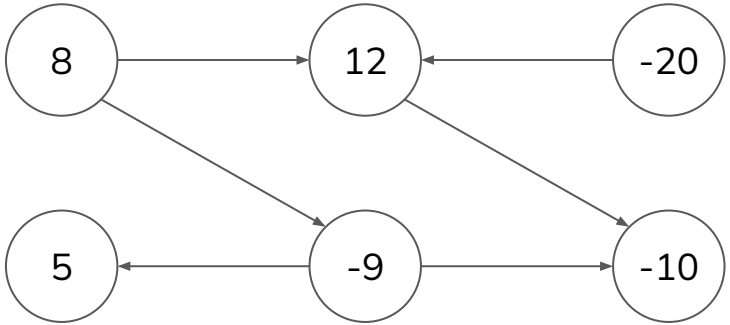
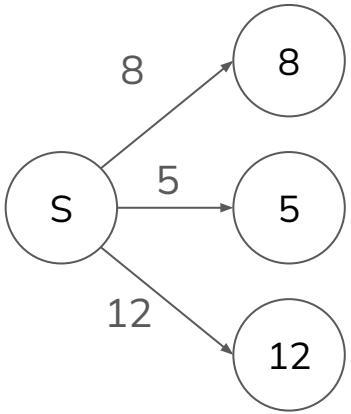


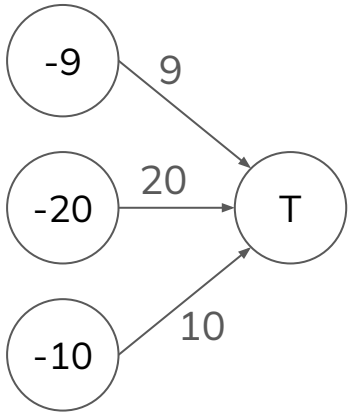
Action: Decide not to pick the positive vertex // Decide to pick the negative vertex.

Modeling Problems with Cut

- Create a vertex in the flow network for each vertex in the graph.
- By default, we choose the positive vertices but not the negative ones:
  - If we decide not to choose a positive vertex: pay a cost. (Build an edge S → u)
  - If we decide to choose a negative vertex: pay a cost. (Build an edge u → T)

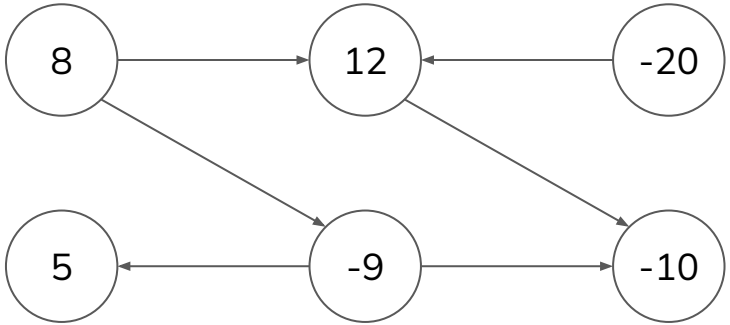


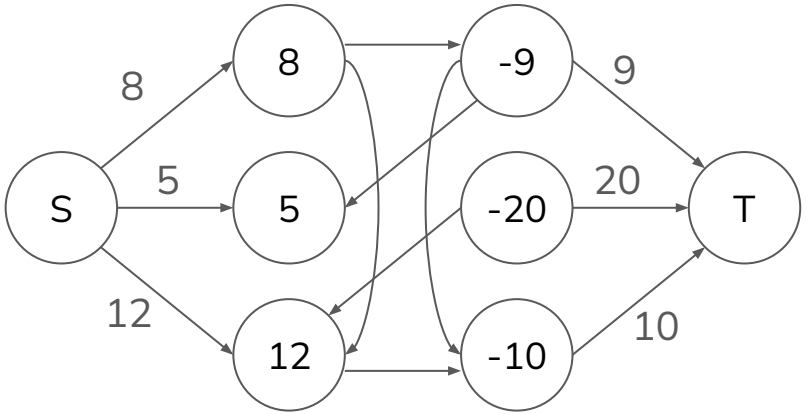




Modeling Problems with Cut

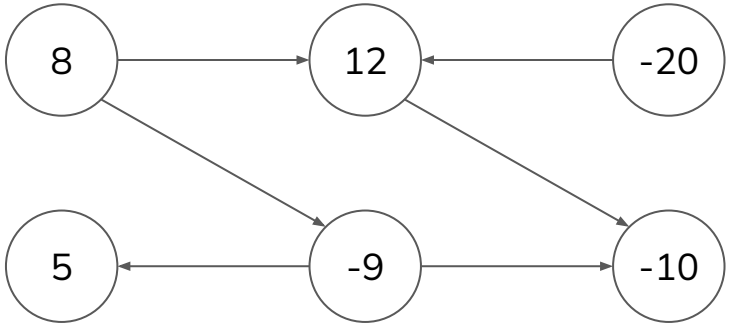
- A **conflict** happens if we picked u but did not pick v for some edge u → v.
  - This means the edges S → u (we picked u) and v → T exist (we did not pick v).
  - Build an edge u → v to form a S-T path (S → u → v → T).
  - What is the cost to "cut" this conflict?

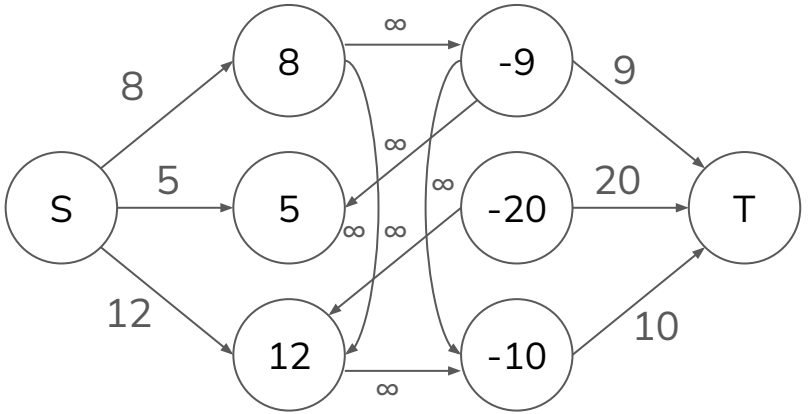




Modeling Problems with Cut

- A **conflict** happens if we picked u but did not pick v for some edge u → v.
  - This means the edges S → u (we picked u) and v → T exist (we did not pick v).
  - Build an edge u → v to form a S-T path (S → u → v → T).
  - What is the cost to "cut" this conflict? Infinity, it's illegal to cut this edge.

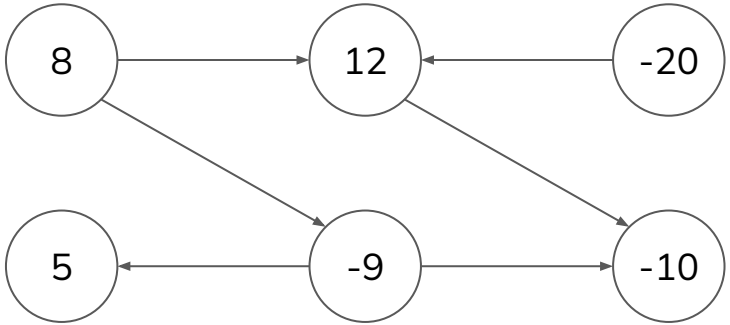


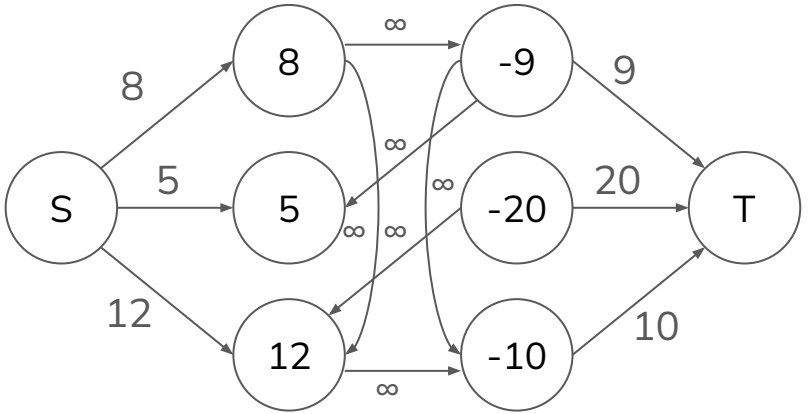


Modeling Problems with Cut

Solution:

- For each vertex with weight w, build the edge S → u with capacity w if w is positive, or build the edge u → T with capacity -w if w is negative.
- For each edge u → v, build the edge u → v with infinite capacity.
- Answer = Minimum Cut = Maximum Flow.





Modeling Problems with Cut

Project Selection Problem: You are given `N` projects and `M` machines. Each project requires a certain set of machines. Project `i`, if completed, generates `a[i]` revenue but machine `j` costs `b[j]` to purchase (it can be used **multiple times**). Find the maximum profit you can make by choosing machines and projects wisely.

Project	Machines	Revenue
1	1, 2	10
2	2, 3	20
3	2, 3, 4	3

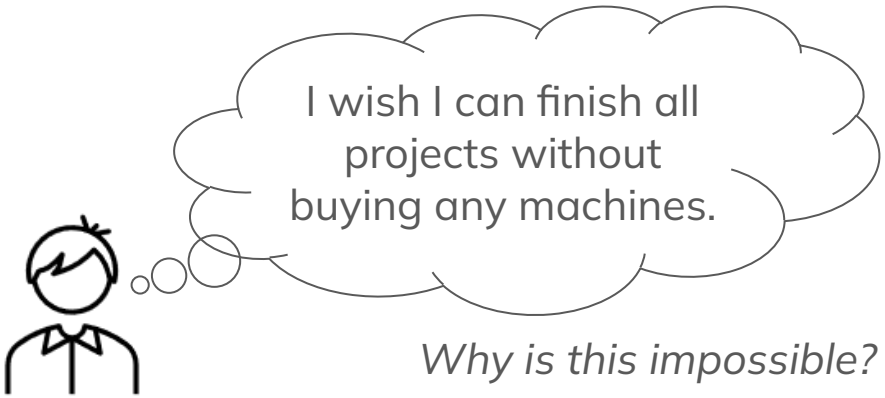
Machine	1	2	3	4
Cost	5	15	4	7

Profit = 10 + 20 - 5 - 15 - 4 = 6

Modeling Problems with Cut

Let's apply "optimistic thinking" again:

What is the ***best ever scenario*** you can wish for?



Why is this impossible?

Project	Machines	Revenue
1	1, 2	10
2	2, 3	20
3	2, 3, 4	3

conflicts!

Machine	1	2	3	4
Cost	5	15	4	7

Modeling Problems with Cut

Let's apply "optimistic thinking" again:

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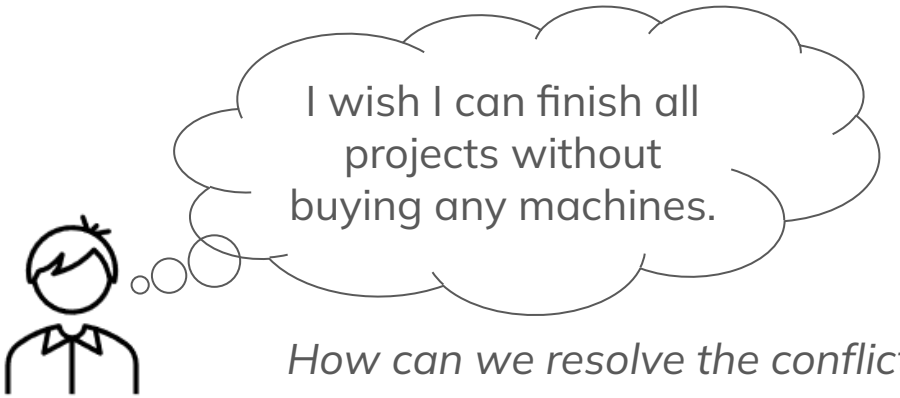


How can we resolve the conflict?

Give up the revenue

Project	Machines	Revenue
1	1, 2	10
2	2, 3	20
3	2, 3, 4	3

Machine	1	2	3	4
Cost	5	15	4	7

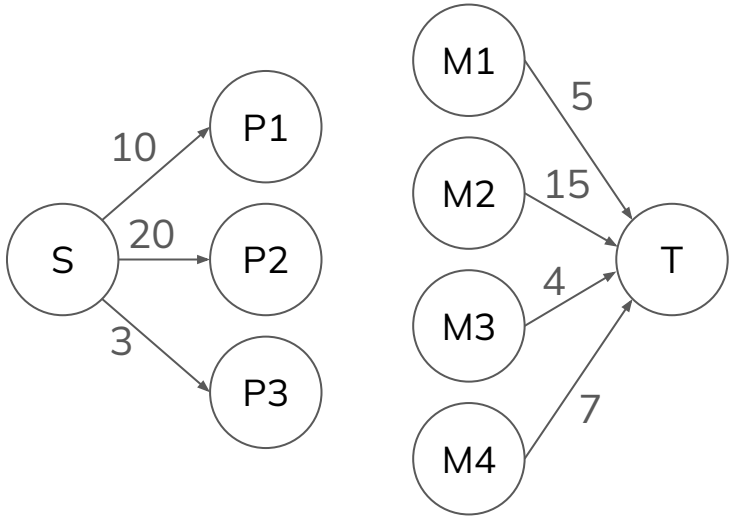
Pay for the machines

Modeling Problems with Cut

- Create a vertex in the flow network for each project and each machine.
- We can choose to abandon a project by paying a cost – create an edge S → p.
- We can choose to buy a machine by paying a cost – create an edge m → T.

Project	Machines	Revenue
1	1, 2	10
2	2, 3	20
3	2, 3, 4	3

Machine	1	2	3	4
Cost	5	15	4	7

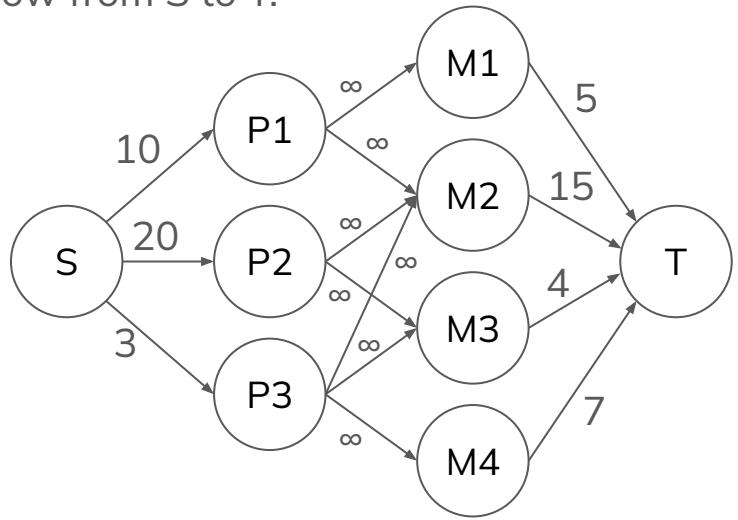


Modeling Problems with Cut

- A **conflict** happens if we chose a project without buying a machine required.
  - For each requirement (project, machine), we can build the edge p → m.
  - If we chose a project (the edge S → p remains) and did not buy the machine (the edge m → T remains), then there is a flow from S to T.

Project	Machines	Revenue
1	1, 2	10
2	2, 3	20
3	2, 3, 4	3

Machine	1	2	3	4
Cost	5	15	4	7



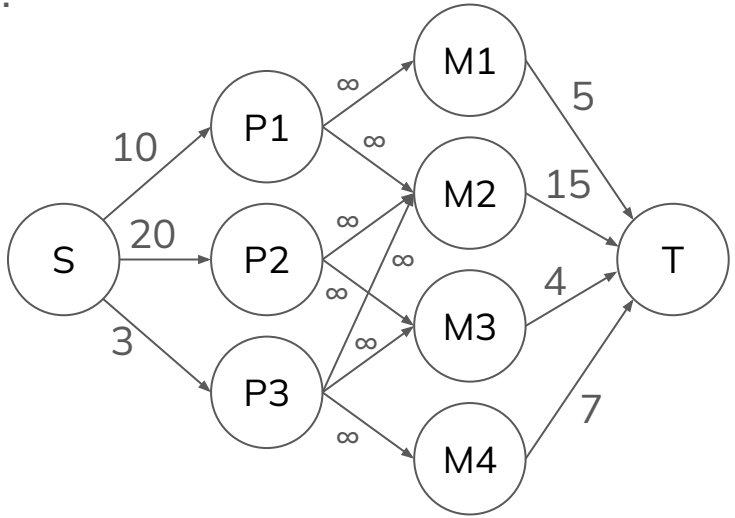
Modeling Problems with Cut

Solution:

- For each project p or machine m, build the edge S → p or m → T (capacity = cost).
- For each requirement (p, m), build the edge p → m with infinite capacity.
- Answer = Minimum Cut = Maximum Flow.

Project	Machines	Revenue
1	1, 2	10
2	2, 3	20
3	2, 3, 4	3

Machine	1	2	3	4
Cost	5	15	4	7



Summary

To solve **minimum cut** problems:

- Apply “optimistic thinking” – what is the ***best ever scenario*** you can get?
- Identify the **conflicts** and the **actions** you can do to resolve these conflicts.
- Model these conflicts/actions (edges) and costs (capacities) into a graph.
- Using **Max-Flow Min-Cut Theorem**, we can run Ford–Fulkerson Algorithm to find the maximum flow (= minimum cut).

Team Problem Set

Team Problem Set

- Try to work on graph modeling!
  - Teams of 3, work collaboratively!
  - Duration: Around 1 hour (depends on lesson time)

- Work on the same question *together*, as opposed to divide & conquer strategies.
  - Time given should be more than enough.
  - Team members should understand each other's solutions.

- Draw the flow network + Describe how to obtain the answer in 1 sentence.

- Teams will be invited to present their solutions to the whole class.