# J264 - Shy Students

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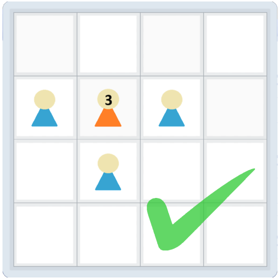
## Background

Problem Idea by `ethening`
Preparation by `bedrockfake, ITO`
Presentation Slides by `ethening, ITO`

**Task:** Design a seating plan for $N$ shy students in a grid.

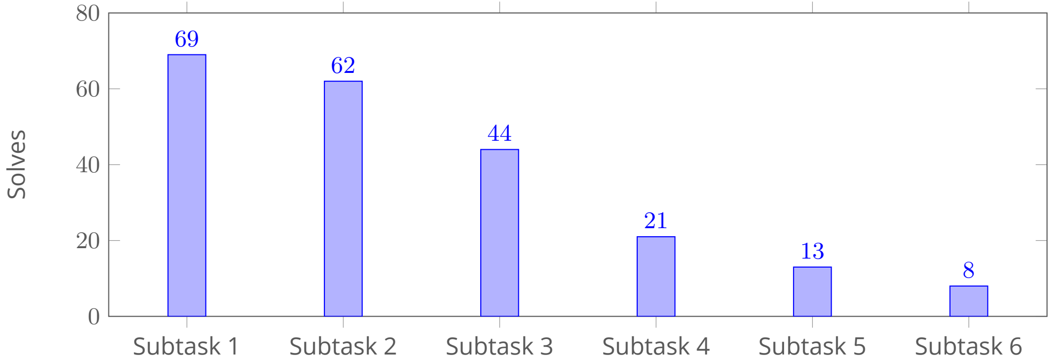
- Each student has shyness $s \in \{1, 2, 3, 4\}$.
- Condition: A student with shyness $s$ can have at most $s$ occupied neighbours.



## Problem Statement

- **Input:** $N_1, N_2, N_3, N_4$ (counts of students with shyness $1, 2, 3, 4$).
- **Output:** A valid grid configuration (dimensions $R \times C$) or `Impossible`.
- **Constraints:**
  - All students must form a single **connected component**.
  - For every student with shyness $s$, degree $\leq s$.
  - $N = \sum N_i \leq 4000$.

## Statistics




First solved by **wy_24215** (Yang Chun Kit) at **57m**.

# Subtasks

| Subtask | Points | Constraints |
|---|---|---|
| 1 | 9 | $N_1 = 0$ |
| 2 | 8 | $N_1 \leq 2$ |
| 3 | 17 | $N_2 = N_3 = 0$ |
| 4 | 19 | $N_3 = 0$ |
| 5 | 20 | $N_2 = 0$ |
| 6 | 27 | No additional constraints |

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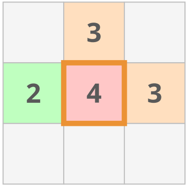
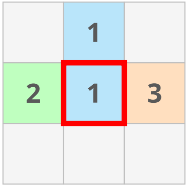
# Definitions

**Degree**

Number of occupied neighbours a cell touches (up, down, left, right).
Constraint: degree $\leq$ shyness $s$.




The **4** has 3 neighbours
degree $= 3 \leq 4$ ✓



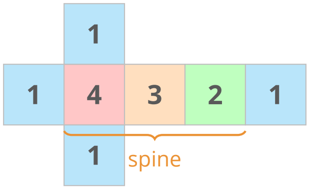

The **1** has 3 neighbours
degree $= 3 > 1$ ✗

# Observation 1: Leaves

**Key Property**

Shyness-1 students can have at most 1 neighbour.
$\rightarrow$ They must be **leaves** in the connectivity graph.

- Two shyness-1 students **cannot be adjacent** (unless $N = 2$). If adjacent, they use up their only connection. They cannot connect to the rest of the class!
- **Structure:** A connected **spine** of non-1 students ($s \geq 2$), with 1s attached as leaves.



# Edge Case: Only Shyness-1 Students

When $N_2 = N_3 = N_4 = 0$:


$N_1 = 1$ — deg $= 0 \leq 1$ ✓

$N_1 = 2$ — deg $= 1 \leq 1$ ✓

$N_1 \geq 3$ — deg $= 2 > 1$ ✗

- Possible iff $N_1 \leq 2$. Output a single row: `1` or `11`.
- From now on, we assume $N_2 + N_3 + N_4 \geq 1$ (there is a spine).

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## Subtask 1: $N_1 = 0$

**Idea**

Only students with shyness $\geq 2$. Simply arrange them in a line!

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | 2 | 3 | 2 | 4 | 3 | 2 |

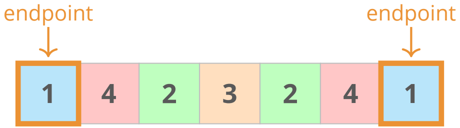
- Interior degree $= 2$. Endpoint degree $= 1$.
- All $s \geq 2$, so degree $\leq s$ is always satisfied.
- This line forms the **spine**.

Expected Score: **9**

## Subtask 2: $N_1 \le 2$

**Idea**

Attach the few shyness-1 students to the **endpoints** of the spine.




- Each endpoint 1 has degree $= 1 \le 1$. ✓
- The spine endpoints now have one extra neighbour, but shyness $\ge 2$ handles it. ✓

Expected Score: **17**

## Subtask 3: $N_2 = N_3 = 0$

There are only 1s and 4s.
With many shyness-1 students, endpoints are not enough.

**Construction**

**3-Row Grid**. Spine in the middle row. Attach 1s **above and below**.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | | 1 | | 1 | | above: 1s |
| 1 | 4 | 4 | 4 | 4 | 4 | 1 | spine: 4s |
| | 1 | | 1 | | | | below: 1s |

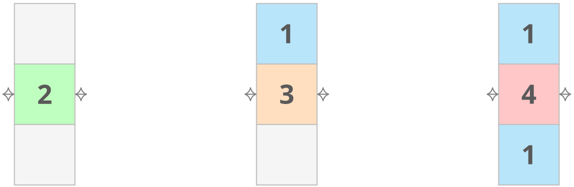
## Subtask 3: $N_2 = N_3 = 0$ (cont.)

- 4s in spine each uses 2 connections for left/right.
- Has spare capacity $4 - 2 =$ **2** (one above, one below).
- **Constraint:** 1s in the same row cannot be adjacent.
- **Solution:** Checkerboard pattern (skip every other column).

Expected Score: **34**

# Capacity Analysis

Each spine student uses 2 connections for the spine. Remaining is **spare capacity**.




spare $= 0$
no room

spare $= 1$
1 above **or** below

spare $= 2$
1 above **and** below

# Capacity Analysis (cont.)

Theoretical maximum 1s:

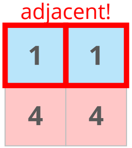
$$\underbrace{2}_{\text{endpoints}} + \underbrace{N_3 \cdot 1}_{\text{from 3s}} + \underbrace{N_4 \cdot 2}_{\text{from 4s}} = 2 + N_3 + 2N_4$$

However, adjacency constraints limit this.

# Constraint: Adjacency of Leaves

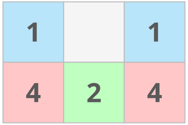
If two 1s are in the same row at consecutive columns, they connect to each other. $\rightarrow$ Degree becomes 2. **Invalid!**

Bad: 1s are adjacent




Each 1 has degree $\geq 2$. ✗

Good: gap between 1s




Each 1 has degree $= 1$. ✓

**Implication:** The order of the spine matters. High-capacity students should be separated to avoid wasting slots.

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# Subtask 4: $N_3 = 0$

**Intuition**

Place a low-capacity student (e.g., **2**) between high-capacity students (e.g., **4**). The 2 acts as a **spacer**, enforcing a gap between the 1s attached to the 4s.

Bad: 4s bunched together

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 1 | | | |
| 4 | 4 | 4 | 2 | 2 | 2 |
| 1 | | 1 | | | |

only 4 extra 1s placed!

Good: alternate 4 and 2

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 1 | | 1 | |
| 4 | 2 | 4 | 2 | 4 | 2 |
| 1 | | 1 | | 1 | |

6 extra 1s placed!

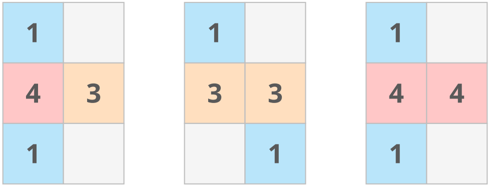
**Strategy:** Alternate 4s and 2s to maximise capacity.

Expected Score: **53**

## Subtask 5: $N_2 = 0$

3s have spare capacity 1 (above OR below).

- **(4, 3) Pair:** 4 takes both slots. 3's slot is blocked. **Waste: 1**.
- **(3, 3) Pair:** Alternate top/bottom. **Waste: 0**.




(4, 3): 3 is blocked (3, 3): both used! (4, 4): 2 wasted!

**Strategy:** Prioritise separating 4s with 3s.

Expected Score: **73**

# Waste Analysis

Waste = Theoretical Capacity - Realisable Capacity

| Pair | Waste | Intuition |
|---|---|---|
| $(4,2)$ | **0** | Best! Perfect spacer. |
| $(3,3)$ | **0** | Alternate above and below. |
| $(3,2),(2,2)$ | **0** | Low capacity, easy to fit. |
| $(4,3)$ | **1** | Okay. 1 slot blocked. |
| $(4,4)$ | **2** | Worst! 2 slots blocked. |

# Optimal Ordering Strategy

**Priority**

Construct the spine in this order to minimize waste:
**(4, 2)** $\rightarrow$ **(4, 3)** $\rightarrow$ Remainder.

- **Why?** Pair high-capacity (4) with spacers (2) first.
- If no 2s left, use 3s as imperfect spacers.
- Avoid (4, 4) adjacency at all costs (Waste: 2).

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## Impossibility Condition

Calculate max placeable 1s:

$$\text{Capacity} = 2 + N_3 + 2N_4 - L$$

where $L$ is total waste from the optimal ordering.

### Condition

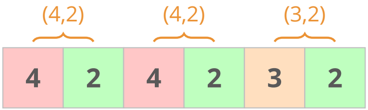
**Impossible** if:

- Only 1s exist and $N_1 \geq 3$.
- $N_1 >$ Capacity.

# Step 1: Build the Spine

Example: $N_1 = 3, N_2 = 3, N_3 = 1, N_4 = 2.$

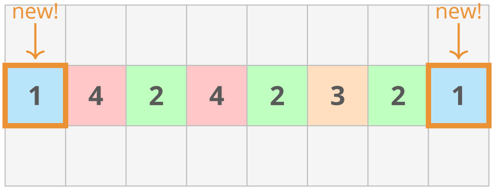
**Order:** $(4,2) \to (4,2) \to (3,2).$




$L = 0$. Remaining $N_1 = 3$.

## Step 2: Endpoints

Place 1s at left/right ends.




Placed 2. Remaining $N_1 = 1$.

# Step 3: Fill Spare Capacity

Greedy Strategy:

- Scan spine left-to-right.
- For each student with spare capacity:
- Check **above**, then **below**.
- Valid if left neighbor in that row is empty.


Position 1 is a **4** (spare $= 2$).

Left cell in row 0 is empty
$\rightarrow$ place above!

$N_1 = 0$. Done!

## Algorithm Summary

1. **Edge Case:** Only 1s? Check $N_1 \leq 2$.
2. **Spine:** Construct using priority $(4,2) \to (4,3) \to \dots$
3. **Endpoints:** Add 1s to spine ends.
4. **Fill:** Add remaining 1s above/below spine where valid.
5. **Check:** If $N_1$ remains, `Impossible`.

Time Complexity: $O(N)$

Expected Score: **100**

## Key Takeaways

- **Change your perspective:** The shyness-1 constraint forces a spine-and-leaf structure. Identifying this early turns the problem from ”how to create a valid construction” to ”how to maximize placement of 1s”.
- **Subtasks as scaffolding:** The subtasks reveal the priorities of the pairings. Build your solution layer by layer, following the guidance of the subtasks.

# Appendix: Proof of Optimality

We prove that the straight-line spine construction maximises the number of shyness-1 students placed.

Let $K = N_2 + N_3 + N_4$. In this appendix, assume $K \geq 1$ (the $K = 0$ case is handled separately).

# Lemma 1: Degree-Budget Upper Bound

Take any valid placement. Let $H$ be the subgraph induced by non-1 students.

- $H$ is connected (all students are connected, and every shyness-1 student is a leaf).
- Let $m = |E(H)|$. Since $H$ is connected on $K$ vertices, $m \geq K - 1$.
- Total degree budget contributed by non-1 students is

$$B = 2N_2 + 3N_3 + 4N_4.$$

Every edge of $H$ consumes 2 budget units, so at most $B - 2m$ units remain for edges from non-1 students to shyness-1 students. Therefore

$$N_1 \leq B - 2m \leq B - 2(K-1) = 2 + N_3 + 2N_4.$$

# Upper Bound: Waste from Adjacency

Not all spare capacity is realisable. For two non-1 cells that are edge-adjacent or corner-adjacent, the union of their grid neighbours has size at most 6.

**Waste per pair**

Two such non-1 cells with values $a$ and $b$ have unavoidable loss $w(a,b) = \max(0,\ a+b-6)$.

Only $(4,4) \to 2$ and $(4,3) \to 1$ produce non-zero waste; all other pairs: $0$.

## Upper Bound: Waste from Adjacency

- Consider any partition of the non-1 cells into pairs of edge/corner-adjacent cells, with at most one unpaired cell.
- For each pair $(a_i, b_i)$, at least $w(a_i, b_i)$ capacity is lost, so total loss is at least $\sum_i w(a_i, b_i)$.
- Minimising this over all such pairings gives a universal lower bound on loss; denote it by $L_{\text{min}}$.

Combining with Lemma 1:

$$N_1 \leq 2 + N_3 + 2N_4 - L_{\text{min}}.$$

# Lower Bound: Construction Achieves the Bound

The straight-line spine on a 3-row grid achieves

$$N_1 = 2 + N_3 + 2N_4 - L_{\text{min}}.$$

1. **Maximal spare.** The spine is a path on $K$ non-1 cells, so it uses exactly $K - 1$ non-1 edges, hence spare budget is exactly $2 + N_3 + 2N_4$.
2. **Exact loss.** Along this spine, loss is exactly the sum of pair losses from its adjacent pairs; with ordering $(4, 2) \to (4, 3) \to$ remainder this equals $L_{\text{min}}$.
3. **Realisability.** Place 1s greedily in checkerboard pattern above/below the spine. No two placed 1s are adjacent, and every non-wasted slot is filled.

Hence the upper bound is met with equality, so the construction is **optimal**. □

# Computing $L_{\text{min}}$: Optimal Pairing

Greedy strategy to minimise total waste:

1. **Pair 4s with 2s** (waste 0). Uses $\min(N_4, N_2)$ of each.
2. **Pair remaining 4s with 3s** (waste 1 each).
   Let $q = \min\big(\max(0,\ N_4 - N_2),\ N_3\big)$.
3. **Pair remaining 4s together** (waste 2 each).
   Let $r = \left\lfloor \dfrac{\max(0,\ N_4 - N_2 - N_3)}{2} \right\rfloor$.

$$L_{\text{min}} = q + 2r$$

**Final Answer**

Impossible $\iff N_1 > 2 + N_3 + 2N_4 - L_{\text{min}}$.
Edge case: if $K = 0$ (only 1-students), possible iff $N_1 \le 2$.