Recursion, Divide & Conquer

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Hong Kong Olympiad in Informatics

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Functions

- Functions in Math:
  - $f(x) = \sin(x)$, $f(x) = x^2 + 2x + 1$
  - Substitute x (Input) into the f() (Process), return the calculated value (Output)

- Functions in OI
  - Similar to maths
  - As a subroutine that process the input/parameters and return the outputs

Function in OI

```
int f(int x) {
  int y = x * x + x + 2;
  return y;
}
int main() {
  cout << f(7) << endl;
}
```

Define a function name and input

Process the input

Return the output

Call the function with parameter x = 7

Function in OI

- There can be multiple input for a function

```
double dist (double x1, double y1, double x2, double y2) {
 return sqrt((x1 - x2) * (x1 - x2) + (y1 - y2) * (y1 - y2));
}
```

Procedure

- Similar to functions
- It can take inputs and process like functions
- But with no return values

Procedure

```
void hello(int n) {
    for (int i = 1; i <= n; i++)
        cout << “HKOI” << endl;
}
int main() {
    hello(6);
}
```

Define a function name and input

Process the input

Call the procedure with n = 6

Recursion

- A function or procedure that call itself

```
int gcd(int a, int b) {
    if (b == 0) return a;
    return gcd(b, a % b);
}
```

Recursion

```
int main() {
      cout << gcd(42, 4);
}
```

Recursion

```
int main() {
      cout << gcd(42, 4);
}
```

```
(a = 42,b = 4)
int gcd(int a, int b){
      if (b == 0) return a;
      return gcd(b, a % b);
}
```

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Recursion

```
int main() {
    cout << gcd(42, 4);
}
                        (a = 42,b = 4)
            int gcd(int a, int b){
                if (b == 0) return a;
                return gcd(b, a % b);
            }                          (a = 4,b = 2)
                          int gcd(int a, int b){
                              if (b == 0) return a;
                              return gcd(b, a % b);
                          }
```

Recursion

```
int main() {
    cout << gcd(42, 4);
}
```

(a = 42,b = 4)

```
int gcd(int a, int b){
    if (b == 0) return a;
    return gcd(b, a % b);
}
```

(a = 4,b = 2)

```
int gcd(int a, int b){
    if (b == 0) return a;
    return gcd(b, a % b);
}
```

(a = 2,b = 0)

```
int gcd(int a, int b){
    if (b == 0) return a;
    return gcd(b, a % b);
}
```

Recursion

```
int main() {
    cout << gcd(42, 4);
}
```

(a = 42,b = 4)

```
int gcd(int a, int b){
    if (b == 0) return a;
    return gcd(b, a % b);
}
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(a = 4,b = 2)

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int gcd(int a, int b){
    if (b == 0) return a;
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(a = 2,b = 0)

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int gcd(int a, int b){
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Recursion

```
int main() {
    cout << gcd(42, 4);
}
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(a = 42,b = 4)

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int gcd(int a, int b){
    if (b == 0) return a;
    return gcd(b, a % b);
}
```

(a = 4,b = 2)

```
int gcd(int a, int b){
    if (b == 0) return a;
    return 2;
}
```

(a = 2,b = 0)

```
int gcd(int a, int b){
    if (b == 0) return 2;
    return gcd(b, a % b);
}
```

Recursion

```
int main() {
    cout << gcd(42, 4);
}
```

```
(a = 42,b = 4)
int gcd(int a, int b){
    if (b == 0) return a;
    return 2;
}
```

```
(a = 4,b = 2)
int gcd(int a, int b){
    if (b == 0) return a;
    return 2;
}
```

```
(a = 2,b = 0)
int gcd(int a, int b){
    if (b == 0) return 2;
    return gcd(b, a % b);
}
```

Recursion

```
int main() {
      cout << 2;
}
                          (a = 42,b = 4)
                 int gcd(int a, int b){
                       if (b == 0) return a;
                       return 2;
                 }                             (a = 4,b = 2)
                                int gcd(int a, int b){
                                      if (b == 0) return a;
                                      return 2;
                                                              (a = 2,b = 0)
                                }
                                                    int gcd(int a, int b){
                                                          if (b == 0) return 2;
                                                          return gcd(b, a % b);
                                                    }
```

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Recursion

- By using recursion, we can simplify our code

- Recursion can help us solve problem with following properties:
  - The problem can be divided/reduced into same problem with smaller parameter
  - We need informations of the sub-problems to solve the current one

Recursion

```
int gcd(int a, int b) {
    if (b == 0) return a;
    return gcd(b, a % b);
}
```

Base case: stop the recursion

Recurrence relation: to divide the current problem into a sub-problem

Example – Fibonacci number

- Find the n-th Fibonacci number
- Warning:
- **DO NOT** use recursion to calculate the n-th Fibonacci number using recursion without memoization
- Base case: $F_0 = 0$, $F_1 = 1$
- Recurrence relation: $F_n = F_{n-1} + F_{n-2}$

Example — Fibonacci number

- Find the n-th Fibonacci number
- Base case: $F_0 = 0$, $F_1 = 1$
- Recurrence relation: $F_n = F_{n-1} + F_{n-2}$

```
int fib(int n) {
    if(n == 0 || n == 1) return n;
    return fib(n - 1) + fib(n - 2);
}
```

Example – Tower of Hanoi

- There are 3 pegs, numbering 0, 1, 2 from left to right
- There are initially N disks of different size on the 0th peg.
- The disks increase in size from top to bottom
- The goal is to move entire tower of disks from 0th peg to the one of the other peg
- One disk can be moved from the top of one peg to another empty peg or peg with larger size topmost disk

Example – Tower of Hanoi

- Strategy:
- Name each peg by start, end and intermediate
- Move the topmost n-1 disks from start to intermediate
- Move the largest disk to end, now the start peg is empty
- Treat the start peg as intermediate peg, intermediate peg as start and repeat the step above.

Example – Tower of Hanoi

Image source:

https://medium.com/@jamalmaria111/tower-of-hanoi-js-algorithm-3f667fa46f0f

STEP1
N Disks
move N-1
Recusively
Source
Auxiliary
Destination

STEP2
last disk
N-1
Source
Auxiliary
Destination

STEP3
N-1
last disk
Source
Auxiliary
Destination

STEP4
move N-1
Recusively
Source
Auxiliary
Destination

Example – Tower of Hanoi

- Base case: When n = 1, move the disk from start to end.
- Recurrence relation, when n > 1:
  - Move n - 1 disk from start to intermediate
  - Move 1 disk from start to end
  - Move n - 1 disk from intermediate to end

- Let P(n, start, inter, end) represents the problem we want to solve. Where we move n disks from start peg to end peg using the intermediate peg.
- If we follow this strategy, it will be the minimum steps possible which requires 2^n - 1 steps.

Pseudocode

```
void p(int n, int start, int inter, int end) {
    if(n == 1) {
        [Move largest disk from start to end]
    } else {
        P(n - 1, start, end, inter)
        [Move largest disk from start to end]
        P(n - 1, inter, start, end)
    }
}
```

Exhaustion

- Sometimes we don't know fast algorithms to solve a problem
- Exhaust all possible state/solutions
- Check whether the state is the one we want
- Output the best / first found

Example – Subset sum

- Given a list of N integers, find a subset of integers that sums to S.
- e.g. A = [1, 2, 4, 8, 16], S = 13
- The subset that gives S = 13 is [1, 4, 8]

Example – Subset sum

- Try subset of size 1, 2, 3, … , n
- Using for loop?

Example – Subset sum

- What a mess and we need so many lines of repetitive code.

```
for(int i = 0; i < n;i++)
      if (a[i] == S) cout << a[i] << endl;
for(int i = 0; i < n; i++)
      for(int j = i + 1; j < n; j++)
            if(a[i] + a[j] == S) cout << a[i] << ‘ ‘ << a[j] << endl;
for(int i = 0; i < n; i++)
      for(int j = i + 1; j < n; j++)
            for(int k = j + 1; k < n; k++)
                  if(a[i] + a[j] + a[k] == S) cout << a[i] << ‘ ‘ << a[j] <<  ‘ ‘ << a[k] << endl;
```

Example – Subset sum

- This can be solved by recursion
- For each number, we can decide whether or not to choose it.
- Try both
- Define F(i, X) denoting we have considered i number and the current sum is X
- Base case: i == n, if X = S, output, else this subset can't add up to the sum we want.
- Recurrence relation:
  - Try F(i + 1, X + a[i]) // Include this number
  - and F(i + 1, X) // Not to include this number

Example – Subset sum

```
bool choose[n]; // all initialized to false
void solve(int i, int X) {
      if(i == n && X == S) {
            for (int j = 0; j < n; j++)
                  if(choose[j]) cout << a[j] << ' ';
      } else if(i == n)
            return;
      else {
            choose[i] = true; solve(i + 1, X + a[i]);
            choose[i] = false; solve(i + 1, X);
      }
}
```

Example – Subset sum

- Time complexity: $O(2^N)$
- Which can give solutions within 1s when N <= 20
- There exist solutions with pseudo-polynomial time by using DP

Example – Permutation

- Generating sequences of permutation
- Can also be done using next_permutation in STL
- Time complexity: O(N!)
- Useful when N <= 10

Example – Permutation

- Suppose we want to permutate the string S = “ABCD”
- Define permutation(S, pos) as permuting S while the current position at pos
- Base case: When pos == S.size(), we can’t swap anymore so we just output
- Recurrence relation: For all i > pos, swap S[i] and S[pos], permutation(S, pos + 1)

Example – Permutation

```
void permutation(string s, int pos) {
    if(pos == s.size()) cout << s << endl;
    else {
        for (int i = pos; i < s.size(); i++) {
            swap(s[i], s[pos]); // apply the change
            permutation(s, pos + 1); // recur to the next state
            swap(s[i], s[pos]); // undo the change
        }
    }
}
```

Practice problem

- 01046 One-Step Tower of Hanoi
- 01014 Stamps
- 01031 Permutations
- 01037 Combinations
- 20296 Safecrackers
- 30098 Generating Fast Permutations

Branch & Bound

- We may noticed that some state are invalid and recurring further won’t make it valid again
- We can skip this state and not recur all the state below it which may saves a lot of time

Example – Subset sum revisited

- Current state: F(i, X)
- If X > S, adding any more number or not adding will not sum to S anyway
- Stop the recursion when X > S!

Example – Subset sum revisited

```
bool choose[n]; // all initialized to false
void solve(int i, int X) {
     if(i == n && X == S) {
          for (int j = 0; j < n; j++)
               if(choose[j]) cout << a[j] << ' ';
     } else {
          choose[i] = false; solve(i + 1, X);
          if (X + a[i] > S) return; // Branch cutting
          choose[i] = true; solve(i + 1, X + a[i]);
     }
}
```

Example – Subset sum revisited

- Constant optimization
- Time complexity: $O(2^N)$
- There is no guarantee we always cut the branch, but we can always do so when possible.

Practice problem

- 01049 Chocolate
- 01050 Bin packing
- 20750 8 Queens Chess Problem
- T183 Exam Anti Cheat (You can get at least 50 marks by cutting branch)

break;

Divide & Conquer

- Divide the problem into smaller and independent sub-problems that are the same as the original problem
- If the sub-problem is easy to solve, solve directly. Otherwise divide it into smaller sub-problems recursively.
- Combine the result/solutions from sub-problems to solve the original problem.

Master Theorem (Warning: Maths ahead)

- Master Theorem is used to calculate the time complexity of a divide-and-conquer algorithm
- Assume the time cost function of the problem is T(n)
- If $T(n) = aT(n / b) + O(n^d)$
- $O(n^d)$ can be regarded as the time cost of combining solutions
- $aT(n / b)$ can be regarded as dividing the problem into a sub-problems with parameters n/b.

Master Theorem

- $T(n) = aT(n / b) + O(n^d)$
- If $d > \log_b a$, $T(n) = O(n^d)$
- If $d = \log_b a$, $T(n) = O(n^d \log n)$
- If $d < \log_b a$, $T(n) = O(n^{\log_b a})$

- In real coding, you can just ignore all of the above calculations

Example – Big Mod

- Find B^P % M
- Naive solution: multiply B for P times, doing mod every time.
- Time complexity: O(P)

Example – Big Mod

- Find B^P % M
- When P is even, B^P % M = $B^{P / 2}$ % M * $B^{P / 2}$ % M
- When P is odd, B^P % M = B * ($B^{(P - 1) / 2}$ % M * $B^{(P - 1) / 2}$ % M) % M
- Base case: When P = 0, bmod(b, p, m) = 1 % M
- Recurrence relation:
  - If P is even: bmod(b, p, m) = bmod(b * b % m, p / 2, m)
  - If P is odd: bmod(b, p, m) = b * bmod(b * b % m, p / 2, m), here p / 2 is integer division so we can just skip the -1

Example – Big Mod

```
typedef long long ll;
ll bmod(ll b, ll p, ll m) {
    if(p == 0) return 1 % m;
    else if(p % 2 == 0) return bmod(b * b % m, p / 2, m);
    else return b * bmod(b * b % m, p / 2, m) % m;
}
```

- Super useful, you can just add this to your code template

Example – Big Mod

- Here, we divide P by 2 in each recursion
- There are at most log(P) times before P becomes 0
- Time complexity: O(log P)

- Way faster than O(P)

Example – Big Mod

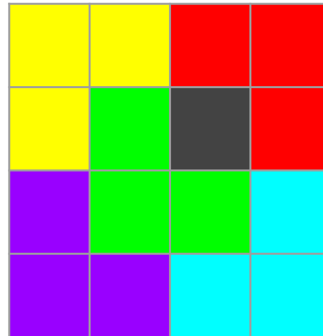
- How can we analyze the time complexity using master theorem?
- At each recursion, we divide problem f(P) into **one** sub-problem f(P / 2)
- It takes O(1) to combine solutions because it is simple multiplication

Example – Big Mod

- $T(n) = aT(n / b) + O(n^d)$
- If $d > \log_b a$, $T(n) = O(n^d)$
- If $d = \log_b a$, $T(n) = O(n^d \log n)$
- If $d < \log_b a$, $T(n) = O(n^{\log_b a})$
- $T(P) = T(P / 2) + O(1)$, $a = 1$, $b = 2$, $d = 0$, $\log_b a = 0$
- We have the case $d = \log_b a$, substitute the number we have $O(\log P)$

Example – L-pieces

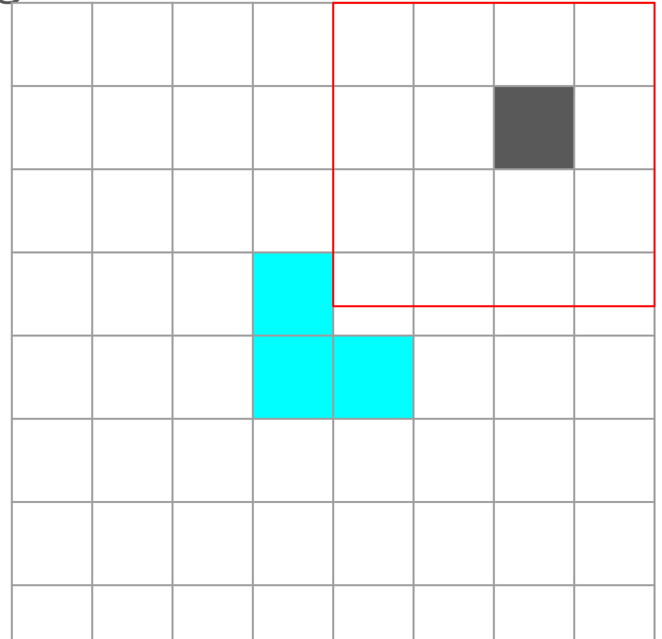
- Given a grid of N * N, all cells are initially empty except one of the cells.
- N is a power of 2
- Fill the grid with L shape
- e.g., a 4 * 4 grid solution, the gray cell is originally filled



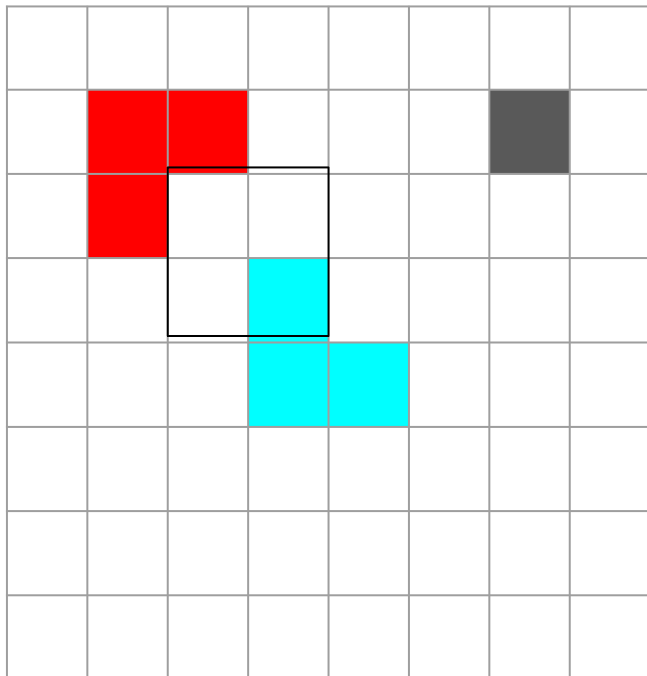
Example – L-pieces

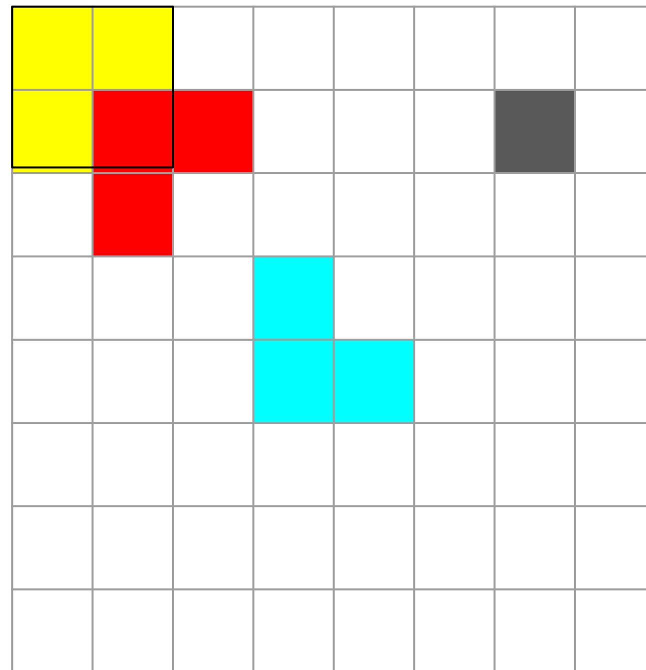
- Idea:
- Divide the grid into 4 regions, each are N / 2 * N / 2
- Put a L piece right next to the corner of the region that contain an filled cell
- Solve recursively for each region
- Base case: 2 * 2 grid, just simply put a L piece

- The gray cell is the given cell
- The cyan L-piece is the one we are putting



Example – L-pieces





Example – L-pieces

- At each step, we divide our problem F(n) into 4 sub-problem F(n / 2)
- Combining the sub-problem cost O(1) as we don't actually have to combine
- $T(n) = 4T(n / 2) + O(1)$
- $a = 4$, $b = 2$, $d = 0$, $\log_b a = 2$
- $T(n) = O(n^{\log_2 4}) = O(n^2)$

Example – Merge sort & no. of inversions

- Merge sort is a well known sorting algorithm that sort an array of length n in O(n log n) time
- This can be extended to count no. of inversions which will be discussed later

Example – Merge sort & no. of inversions

- Given the array A of length n
- Steps:
  - Divide A into two smaller array of length n / 2
  - Sort each of them recursively with merge sort
  - Combine the two sorted array into one array
- Base case:
  - When n = 1, no sorting is needed.
  - When n = 2, easy comparison with an if statement (You can choose to just recur to n = 1)

Example – Merge sort & no. of inversions

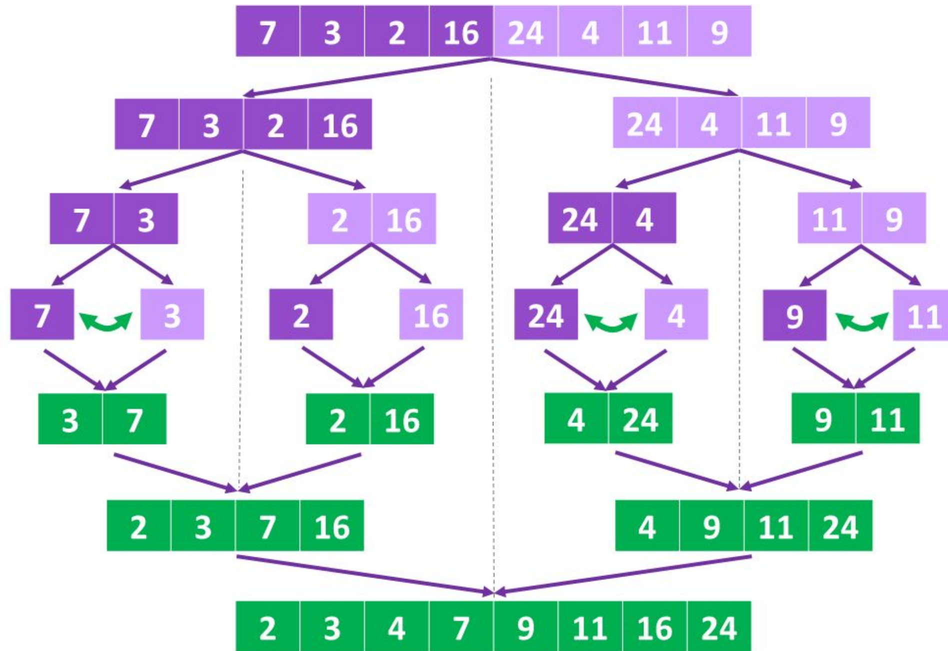


Image Source:

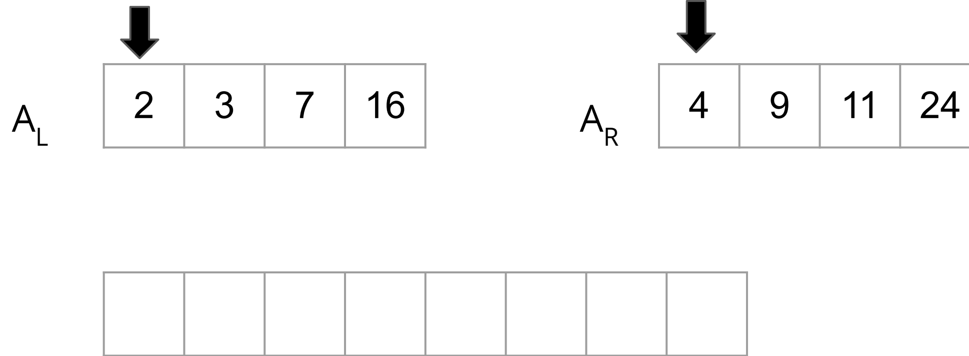
https://www.101computing.net/merge-sort-algorithm/

Example – Merge sort & no. of inversions

- How can we merge the two sorted array?

- Let i be the pointer for the first array A_L
- And j be the pointer for the second array A_R
- Compare $A_L[i]$ and $A_R[j]$
- If $A_L[i] <= A_R[j]$, put $A_L[i]$ into the back of the combined array, increment i
- Else put $A_R[j]$ into the back of the combined array, increment j
- Until either we exhaust all element in A_L or A_R, put all the remaining elements in the other array to the combined array directly

Example — Merge sort & no. of inversions



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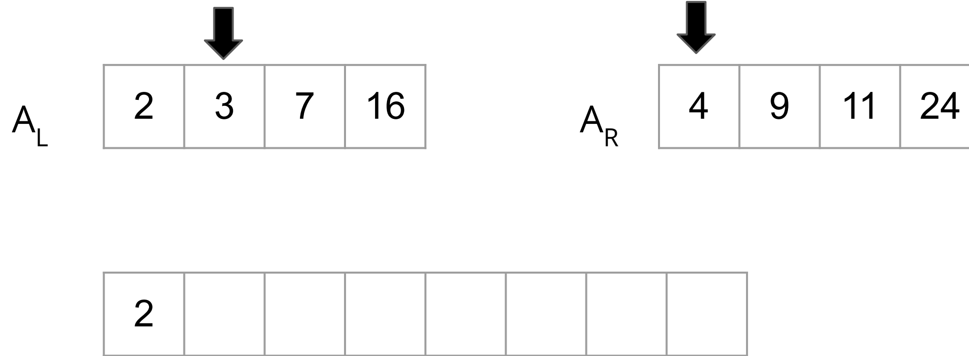

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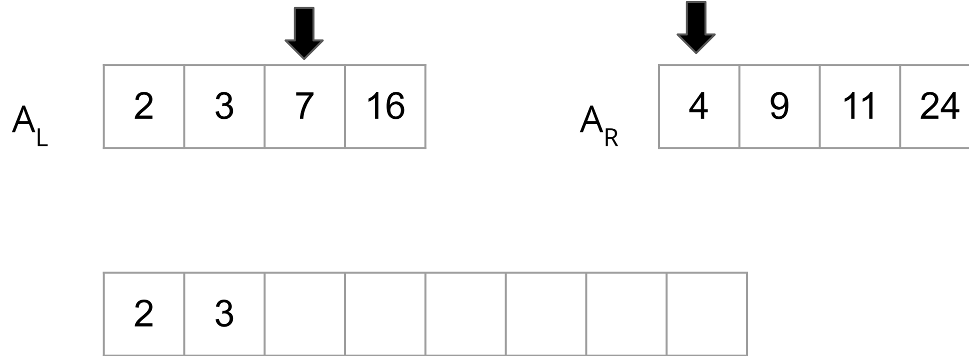

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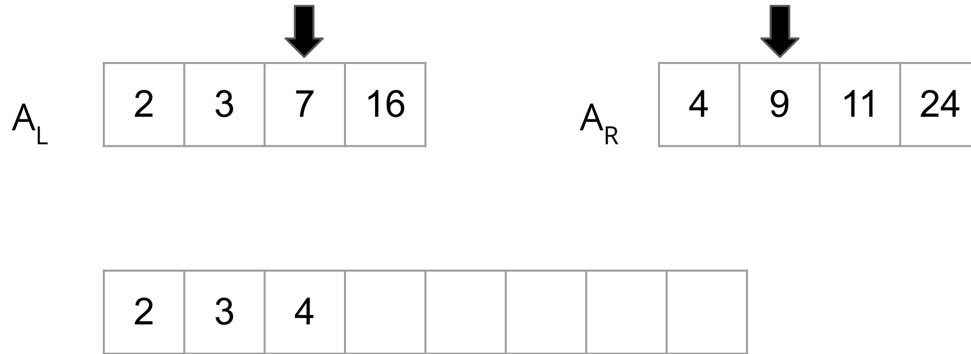
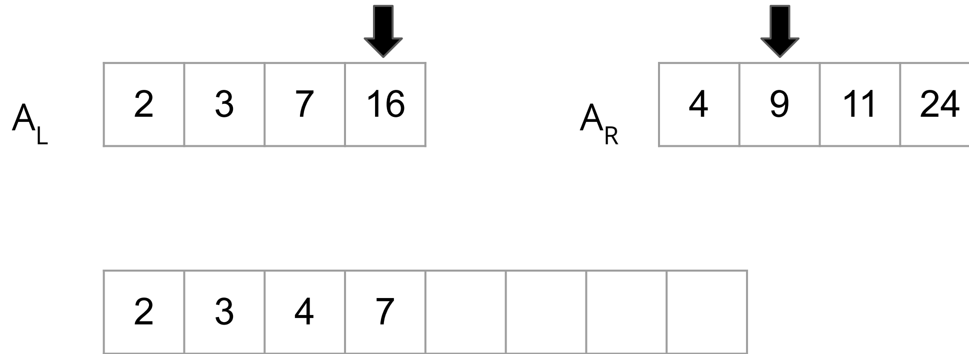


Example – Merge sort & no. of inversions



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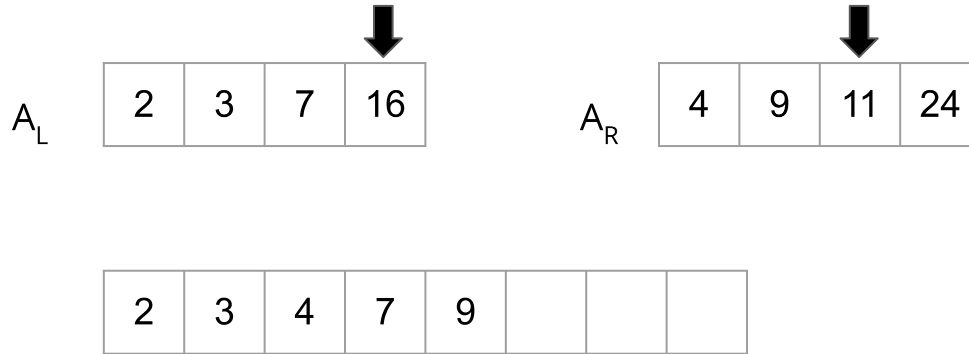

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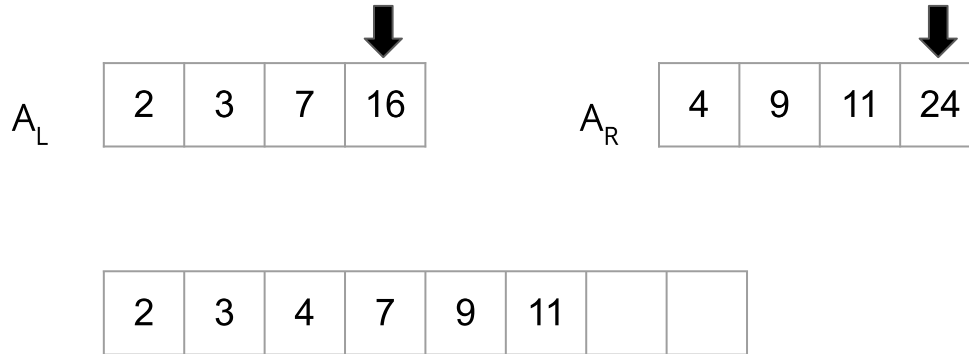
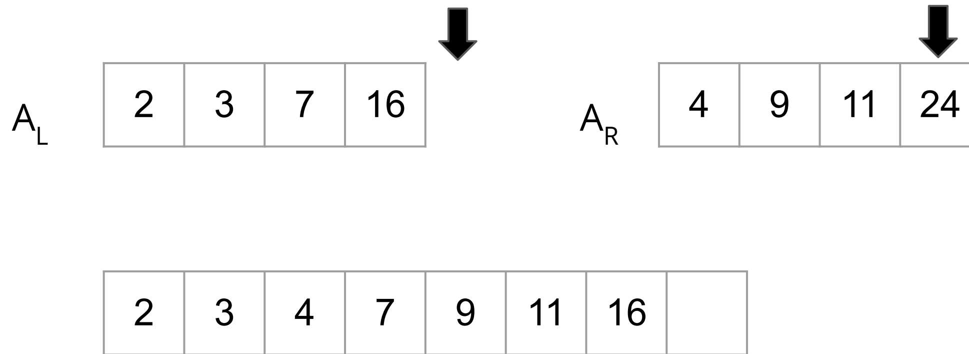
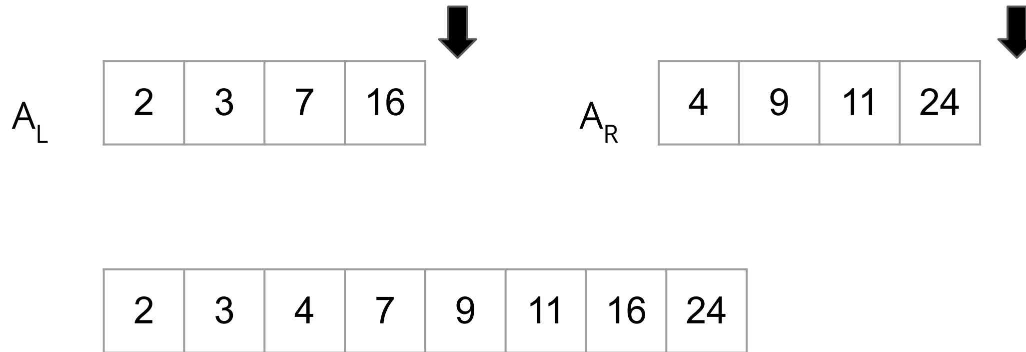


Example – Merge sort & no. of inversions



Example – Merge sort & no. of inversions



Example — Merge sort & no. of inversions

- At each step we divide each problem F(n) into 2 sub-problems F(n / 2)
- Combining the solutions takes O(n)
- $T(n) = 2T(n / 2) + O(n)$
- $a = 2$, $b = 2$, $d = 1$, $\log_b a = 1$
- $d = \log_b a$
- $T(n) = O(n^d \log n) = O(n \log n)$

Example – Merge sort & no. of inversions

Code:

a is the original array

r is the temporary array

s is the starting position

t is the ending position

```
int a[200005], r[200005];
void mergesort(int s, int t) {
  if (s == t)
    return;
  int mid = (s + t) / 2;
  mergesort(s, mid);
  mergesort(mid + 1, t);
  int i = s, j = mid + 1, k = s;
  while (i <= mid && j <= t)
    if (a[i] <= a[j])
      r[k] = a[i], k++, i++;
    else
      r[k] = a[j], k++, j++;
  while (i <= mid)
    r[k] = a[i], k++, i++;
  while (j <= t)
    r[k] = a[j], k++, j++;
  for (int i = s; i <= t; i++)
    a[i] = r[i];
}
```

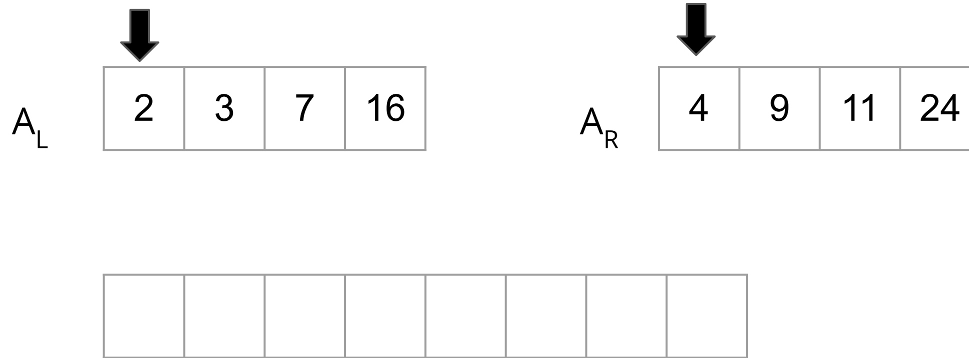
Example – Merge sort & no. of inversions

- Number of inversion of an array of size n is defined as:
- The number of pair (i, j) such that 1 <= i < j <= n and a[i] > a[j]
- e.g. A = [1, 8, 6, 5]
- The inversions are (2, 3), (2, 4) and (3, 4)
- Note that they are indices

Example – Merge sort & no. of inversions

- We can count this number naively using nested for-loop
- Time complexity: $O(n^2)$
- Too slow
- Instead we can modify our merge sort to count this number much faster

Example — Merge sort & no. of inversions

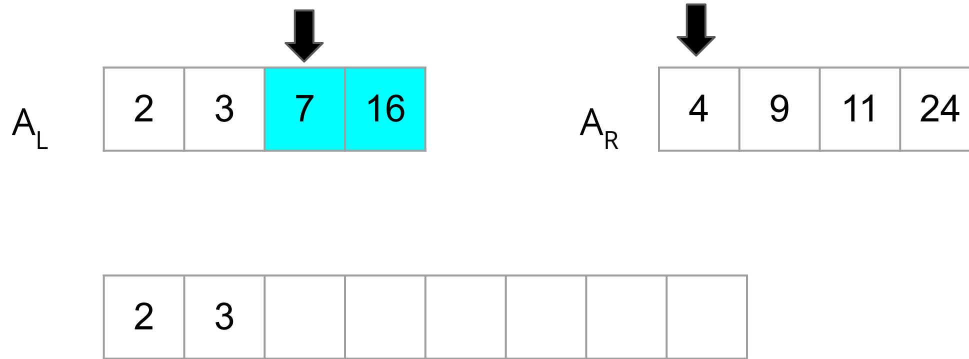


Note that element in A_L is always on the left of A_R.

If at some point $A_L[i] > A_R[j]$, there is an inversion.

Not only one inversion, but actually all elements after $A_L[i]$ including itself are inversions with $A_R[j]$

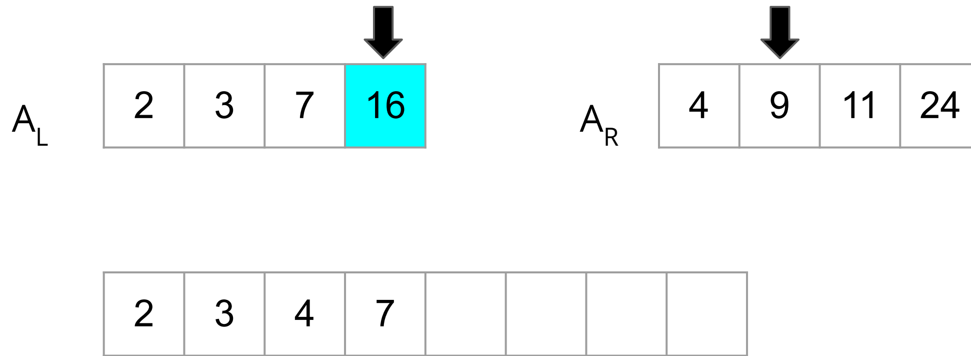
Example – Merge sort & no. of inversions



# of Inversion: 2

Here we encounter the first inversion, there are two elements after.

Example – Merge sort & no. of inversions



# of Inversion: 3

Here we encounter the second inversion, there are only one element after.

Example – Merge sort & no. of inversions

- During the merging process we can count no. of inversions directly in O(1)
- Doing a merge sort takes O(n log n)
- So counting no. of inversions also takes O(n log n)

Example – Merge sort & no. of inversions

Code:

invcnt = # of inversions

Beware that maximum # of inversions = n * (n + 1) / 2

Use **long long** if needed!

```
int a[200005], r[200005];
ll invcnt = 0;
void mergesort(int s, int t) {
  if (s == t)
    return;
  int mid = (s + t) / 2;
  mergesort(s, mid);
  mergesort(mid + 1, t);
  int i = s, j = mid + 1, k = s;
  while (i <= mid && j <= t)
    if (a[i] <= a[j])
      r[k] = a[i], k++, i++;
    else
      invcnt += 1LL * mid - i + 1, r[k] = a[j], k++, j++;
  while (i <= mid)
    r[k] = a[i], k++, i++;
  while (j <= t)
    r[k] = a[j], k++, j++;
  for (int i = s; i <= t; i++)
    a[i] = r[i];
}
```

Practice problem

- 01003 L-pieces
- 01046 One-Step Tower of Hanoi
- 01047 Partially Solved Tower of Hanoi
- M0713 Cream Soda
- J173 Fibonacci Word
- S134 Unfair Santa Claus
- S163 Arithmetic Sequence

Summary

- Recursion is very useful
- Try to write brute force solutions or exhaustion to find patterns and insights
- At least do the brute force subtasks of each questions (if any)
- Divide and Conquer appears a lot in data structures, e.g. Segment Tree
- Recursion also appears a lot in common algorithms, e.g. DFS
- Code more and practice more :)