# Mathematics in OI (II)

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6 April, 2019

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# Main Cast

## $n!$ ($n$ factorial)

Recursive Definition:

- $0! = 1$
- $(n+1)! = (n+1) \times n!$

Combinatorial Definition:

- $n$ is the number of permutations of $1..n$
- $n = 2$: 12, 21
- $n = 3$: 123, 132, 312, 213, 231, 321

# Introducing $\binom{n}{r}$ ($n$ choose $r$)

**Definition**

Let $n$ and $r$ be nonnegative integers. Define $\binom{n}{r}$ (also $C^n_r$) to be the number of ways to choose $r$ elements from $1..n$.
(Sometimes we extend the definition and say $\binom{n}{r} = 0$ for negative $r$.)

Boundary cases:

- $\binom{n}{0} = 1$
- $\binom{n}{n} = 1$
- $\binom{n}{r} = 0$ for $r > n$

Exercise: Verify directly that $\binom{4}{2} = 6$.

## Combinatorial Identities

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \tag{1}$$

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n \tag{2}$$

$$\sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} = (a+b)^n \tag{3}$$

$$\sum_{r=0}^{n} r \times \binom{n}{r} = n \times 2^{n-1} \tag{4}$$

$$\binom{n}{r} = \binom{n}{n-r} \tag{5}$$

We will prove the identities **without using the explicit formula for** $\binom{n}{r}$.

## Equation (1)

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

**What are we counting?**

The number of $r$-element subsets of $1..n$.

- LHS: By definition, it is $\binom{n}{r}$.
- RHS: Condition on whether $n$ is chosen.
  - If no, there are $\binom{n-1}{r}$ ways to choose among the rest
  - If yes, there are $\binom{n-1}{r-1}$ ways to choose among the rest

## Equation (2)

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

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$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

**What are we counting?**

The number of subsets of $1..n$.

- LHS: Condition on the size of the subset. There are $\binom{n}{r}$ subsets of size $r$.
- RHS: By definition, it is $2^n$.

# Equation (3)

$$\sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} = (a+b)^n$$

- If $a$ and $b$ are nonnegative integers, think of both sides as counting the number of length-$n$ sequences consisting of numbers $1..(a+b)$.
- In general, think about the number of ways to form the term $a^r b^{n-r}$ from the expansion of $(a+b)(a+b)...(a+b)$.
- $a = b = 1$:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

- $a = -1, b = 1$:

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$$

## Equation (4)

$$\sum_{r=0}^{n} r \times \binom{n}{r} = n \times 2^{n-1}$$

**What are we counting?**

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$$\sum_{r=0}^{n} r \times \binom{n}{r} = n \times 2^{n-1}$$

**What are we counting?**

From $n$ people, the number of ways to form a committee with a chairperson.
(Yes, you can use everyday language to reason about mathematics!)

- LHS: Form a committee first, then elect chairperson.
- RHS: Choose chairperson first, then pick the rest of the committee.

## Equation (5)

$$\binom{n}{r} = \binom{n}{n-r}$$

**What are we counting?**

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$$\binom{n}{r} = \binom{n}{n-r}$$

**What are we counting?**

The number of $r$-element subsets of $1..n$.

- LHS: By definition, it is $\binom{n}{r}$.
- RHS: Pick $(n-r)$ elements to **exclude** from the subset. There are $\binom{n}{n-r}$ ways to do so.

# Explicit Formula for $\binom{n}{r}$

**Theorem**

For $0 \leq r \leq n$, the following holds:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

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For $0 \leq r \leq n$, the following holds:

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Proofs:

1. Use boundary cases, $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, and induction; OR
2. Show that $\binom{n}{r} r!(n-r)!$ counts the number of permutations of $1..n$.

## A Close Friend: $P^n_r$

**What is $P^n_r$?**

Let $n$ and $r$ be nonnegative integers.
$P^n_r$ counts the number of length $r$ sequences (element order matters!), with each element of $1..n$ appearing at most once.

Boundary cases:

- $P^n_0 = 1$
- $P^n_n = n!$
- $P^n_r = 0$ if $r > n$

Exercise: Verify directly that $P^4_2 = 12$.

# A Close Friend: $P_r^n$

**What is $P_r^n$?**

Let $n$ and $r$ be nonnegative integers.
$P_r^n$ counts the number of length $r$ sequences (element order matters!), with each element of $1..n$ appearing at most once.

Boundary cases:

- $P_0^n = 1$
- $P_n^n = n!$
- $P_r^n = 0$ if $r > n$

Exercise: Verify directly that $P_2^4 = 12$.
Exercise: Prove that, for $0 \leq r \leq n$,

$$P_r^n = \binom{n}{r} \times r! = \frac{n!}{(n-r)!}$$

## Candies and Kids (better known as "Stars and Bars")

### An Arithmetic Question (not really)

Count the number of solutions to $x_1 + \cdots + x_n = S$ where $x_i$ (unknowns) and $S$ (given) are **positive integers**. Denote the answer by $SOL(n, S)$.

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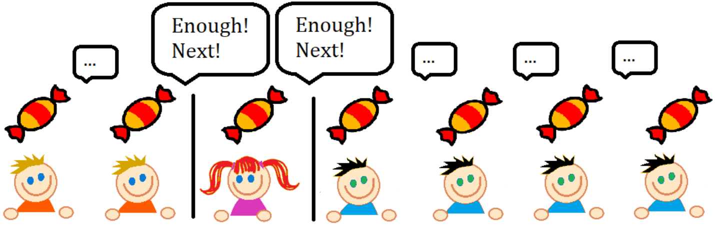

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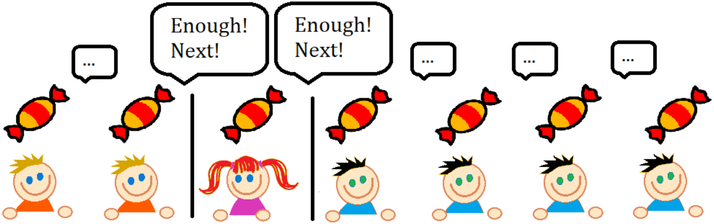


There are $\binom{S-1}{n-1}$ ways to shout "Enough! Next!".
Therefore, $SOL(n, S) = \binom{S-1}{n-1}$.

# A Similar Question

**Another Arithmetic Question (not really)**

Count the number of solutions to $x_1 + \cdots + x_n = S$ where $x_i$ (unknowns) and $S$ (given) are **non-negative integers**. Denote the answer by $SOL2(n, S)$.

# A Similar Question

**Another Arithmetic Question (not really)**

Count the number of solutions to $x_1 + \cdots + x_n = S$ where $x_i$ (unknowns) and $S$ (given) are **non-negative integers**. Denote the answer by $SOL2(n, S)$.

Each child adds one candy to the pool (total becomes $S + n$), then is handed a positive number of candies.
Therefore, $SOL2(n, S) = SOL(n, S + n) = \binom{S+n-1}{n-1}$.

## Computing $\binom{n}{r}$ mod $M$, $n \le 2000$

Use the formula

$$\binom{n}{r} \equiv \binom{n-1}{r} + \binom{n-1}{r-1} \mod M.$$

Code:

```
for (int i = 0; i <= 2000; i++){
  for (int j = 0; j <= i; j++){
    if (j == 0 || j == i)
      ncr[i][j] = 1 % M;
    else
      ncr[i][j] = (ncr[i - 1][j] + ncr[i - 1][j - 1]) % M;
  }
}
```

$O(n^2)$ precomputation, $O(1)$ per query.

## Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Source: Wikipedia

## Computing $\binom{n}{r}$ mod $10^9 + 7$, $n \leq 10^6$

- $10^9 + 7$ can be replaced by any fixed large prime $P$.
- Precompute `fact[k]` $:= k!$ mod $P$ and `inv_fact[k]` $:= (k!)^{-1}$ mod $P$.
- Output `fact[n] * inv_fact[r] * inv_fact[n - r]` mod $P$.

$O(n)$ precomputation, $O(1)$ per query.

## Computing $\binom{n}{r} \bmod M$, $n \leq 10^6$

Exercise: how to compute $\binom{n}{r} \bmod M$ for fixed modulo $M$?

## Computing $\binom{n}{r} \bmod M$, $n \le 10^6$

Exercise: how to compute $\binom{n}{r} \bmod M$ for fixed modulo $M$?
Suggestion:

- Factorize $M$. Write $M = \prod_{i=1}^{k} p_i^{\alpha_i}$.
- Compute quantities similar to `fact[k]` and `inv_fact[k]`, but exclude powers of $p_1, \ldots, p_k$.
- For each $p_i$, compute $\binom{n}{r} \bmod p_i^{\alpha_i}$. (Legendre formula is useful.)
- Use Chinese Remainder Theorem to combine the results.

Exercise: Compute $\binom{6}{4} \bmod 12$ using the method.

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# Fibonacci Numbers (A000045)

**(0, )1, 1, 2, 3, 5, 8, 13, 21, ...**

Recursive Definition:

- $F_0 = 0, F_1 = 1$
- $F_{n+2} = F_n + F_{n+1}$

Combinatorial Definition:

- $F_n$ counts the number of ways to tile a $1 \times (n-1)$ board with squares and dominoes.
- $F_0 = 0$ because, well, a $1 \times (-1)$ board does not exist.

## Computing $F_n$ mod $M$

This identity is useful!

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n \quad (k \geq 1) \tag{6}$$

Putting $k = n$ and $k = n + 1$, respectively, we get

$$\begin{aligned} F_{2n} &= F_n F_{n+1} + F_{n-1} F_n \\ F_{2n} &= F_n F_{n+1} + (F_{n+1} - F_n) F_n \\ \mathbf{F_{2n}} &= \mathbf{2F_n F_{n+1} - F_n F_n} \end{aligned}$$

and

$$\mathbf{F_{2n+1}} = \mathbf{F_{n+1} F_{n+1} + F_n F_n},$$

which give an $O(\log n)$ algorithm to compute $F_n$ mod $M$.

# Equation (6)

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n \quad (k \geq 1)$$

**What are we counting?**

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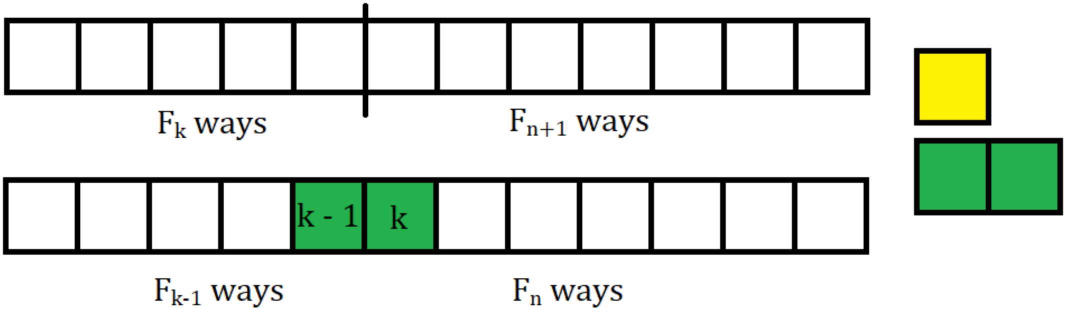
$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n \quad (k \geq 1)$$

**What are we counting?**

The number of square-domino tilings of a $1 \times (n+k-1)$ board.

- LHS: By definition, it is $F_{n+k}$.
- RHS: Condition on whether a domino covers cells $(k-1)$ and $k$.
  - If yes, there are $F_{k-1} \times F_n$ ways to tile the rest.
  - If no, there are $F_k \times F_{n+1}$ ways to tile the rest.

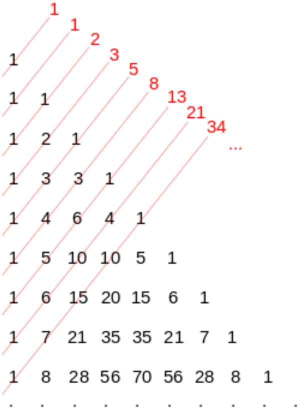
## Illustration



## Relating Fibonacci Numbers and $\binom{n}{r}$

$$F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k} \qquad (7)$$

Note that the last $\frac{n}{2}$ terms are zero.




Source: Wikipedia

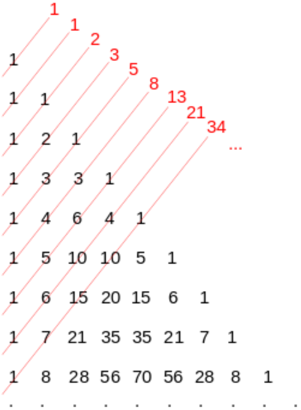
## Relating Fibonacci Numbers and $\binom{n}{r}$

$$F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k} \qquad (7)$$

Note that the last $\frac{n}{2}$ terms are zero.

We are counting the number of square-domino tilings of a $1 \times n$ board.

- LHS: By definition, it is $F_{n+1}$.
- RHS: Condition on the number of dominoes. For tilings with $k$ dominoes, a total of $n-k$ pieces will be used, and so there are $\binom{n-k}{k}$ ways to tile the board.




Source: Wikipedia

## Catalan Numbers (A000108)

Mini-quiz: Do you know the first few Catalan numbers?

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## Catalan Numbers (A000108)

Mini-quiz: Do you know the first few Catalan numbers?

**(1, )1, 2, 5, 14, 42, ...**

Mini-quiz 2: What does this sequence $C_n$ describe?

**Many Things!**

- $C_n$ is the number of valid bracket sequences of length $2n$.
- $C_n$ is the number of triangulations of a convex $(n+2)$-gon.
- $C_n$ is the number of “ordered” rooted binary trees with $n$ nodes.
- The list goes on...

## Let's Take A Look At Small $n$'s

**Definition**

Define $C_n$ to be the number of valid bracket sequences of length $2n$.

Valid means the ( and ) brackets form matching pairs.

- $n = 0$: (empty sequence)
- $n = 1$: ()
- $n = 2$: ()(), (())
- $n = 3$: ()()(), ()(()), (())(), (()()), ((()))
- $n = 4$: ()()()(), ()()(()), ()(())(), ()(()()), ()((())), (())()(), (())(()), (()())(), ((()))(), (()()()), (()(())), ((())()), ((()())), (((())))

## Recurrence Formula

$$C_n = \sum_{k=1}^{n} C_{k-1} \times C_{n-k} \tag{8}$$

**What are we counting?**

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$$C_n = \sum_{k=1}^{n} C_{k-1} \times C_{n-k} \tag{8}$$

**What are we counting?**

The number of valid bracket sequences of length $2n$.

- LHS: By definition, it is $C_n$.
- RHS: Condition on the position of ) which closes the first bracket (. For $1 \le k \le n$, there are $C_{k-1} \times C_{n-k}$ valid bracket sequences with ) positioned at $2k$ (note that the position must be even).

This gives an $O(n^2)$ algorithm to compute $C_n \bmod M$.

## Explicit Formula For $C_n$

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{9}$$

Reformulate the Catalan model as follows:

- You start from $(0, 0)$ and want to reach $(2n, 0)$.
- From $(x, y)$ you can move “up” to $(x+1, y+1)$ or “down” to $(x+1, y-1)$.
- $C_n$ is the number of “good” paths – paths that do not go below the x-axis.

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Reformulate the Catalan model as follows:

- You start from $(0, 0)$ and want to reach $(2n, 0)$.
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- $C_n$ is the number of "good" paths – paths that do not go below the x-axis.

Ideas:

- We count, instead, the number of bad paths.
- Then $C_n = \binom{2n}{n} -$ (number of bad paths).
- Therefore, for the formula to hold, we expect there to be $\binom{2n}{n+1}$ bad paths.

## Reflection Trick

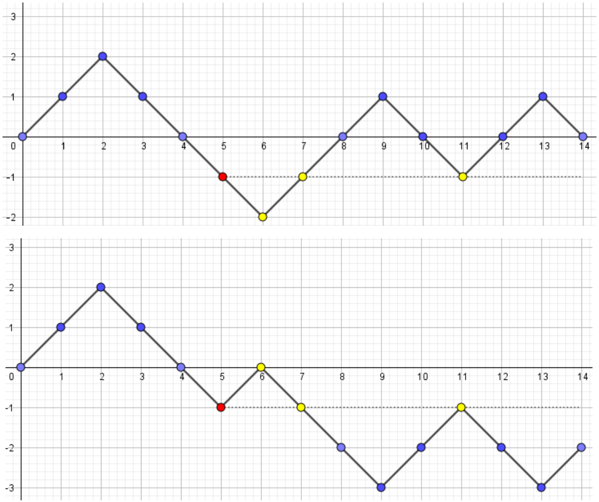
Establish a bijection between:

- Bad paths from $(0, 0)$ to $(2n, 0)$, and
- Paths from $(0, 0)$ to $(2n, -2)$

Idea: Take the portion of the path after the first visit below x-axis. Reflect it about $y = -1$.

There are $\binom{2n}{n+1}$ paths from $(0, 0)$ to $(2n, -2)$. Hence, there are $\binom{2n}{n+1}$ bad paths from $(0, 0)$ to $(2n, 0)$.

# Illustration



# Exercise: Codeforces 26D

Essentially, you are to count the number of good paths from $(0, k)$ to $(n+m, k-m+n)$.

- If $k-m+n<0$, there are no good paths.
- Otherwise, use reflection trick to count the number of bad paths.

Example: $(n, m, k) = (5, 3, 1)$.

Further Reading 1: La Salle-Pui Ching Programming Challenge 2017 Problem L (Let Me Count The Ways) and editorial

Further Reading 2: Young Tableaux

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## Set Union, Set Intersection

**Definition**

Given two sets $A$ and $B$.
Their union $A \cup B$ is the set of elements belonging to *at least one* of $A$ and $B$.
Their intersection $A \cap B$ is the set of elements belonging to *both* of $A$ and $B$.

Example 1
$A := \{\text{Alex}, \text{Anson}, \text{Jason}\}$, $B := \{\text{Alex}, \text{Sampson}, \text{RB}\}$.
Then $A \cup B = \{\text{Alex}, \text{Anson}, \text{Jason}, \text{Sampson}, \text{RB}\}$; $A \cap B = \{\text{Alex}\}$.

Example 2
$A := \{\text{prime numbers}\}$, $B := \{\text{even numbers}\}$
Then $A \cap B = \{2\}$.

Mental Shortcut: intersection = AND, union = OR

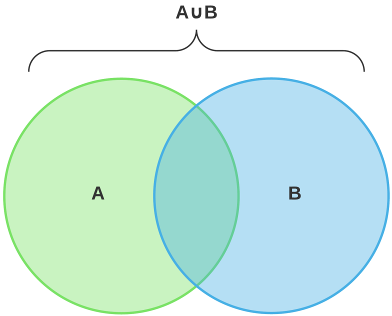
## Subset, Complement

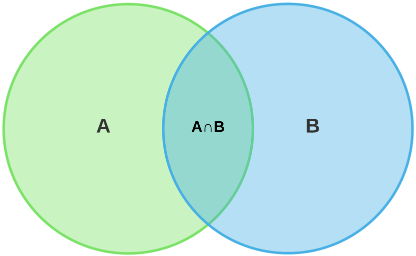
**Definition**

Given two sets $X$ and $Y$.
$Y$ is said to be a subset of $X$ if all elements of $Y$ can be found in $X$.
We write $Y \subseteq X$.
If $Y \subseteq X$, define its complement (with respect to $X$) $X \backslash Y$ to be the set of elements in $X$ but not in $Y$.
We also use $Y^c$ to denote $X \backslash Y$ when it is clear what $X$ is.

Example
Let $X := \{\text{all positive integers}\}, Y := \{1, 4, 9, 16, \dots\}$.
Then $Y \subseteq X$ and $X \backslash Y = \{\text{positive integers that are not perfect squares}\}$.

# Venn Diagram (for two sets)




Source: https://www.lucidchart.com/blog/venn-diagram-symbols-explained

## Inclusion-Exclusion (for two sets)

Generic form:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

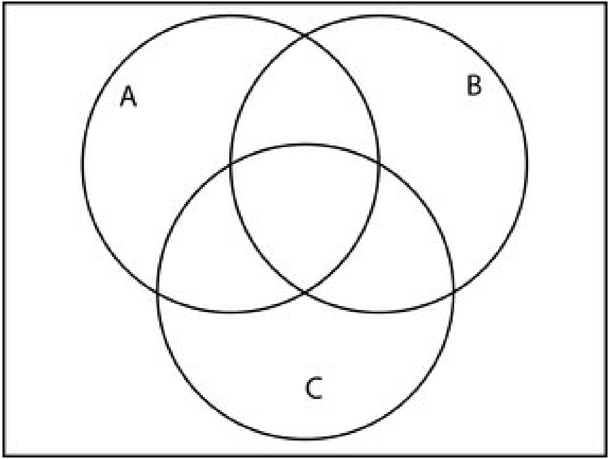
- Answers “how many elements satisfy at least one of the conditions?”
- Useful when **satisfaction of specific conditions** is easier to count

Alternative form:

$$|A \cap B| = |X| - |A^c| - |B^c| + |A^c \cap B^c|$$

- Answers “how many elements satisfy both conditions?”
- Useful when **violation of specific conditions** is easier to count

# Venn Diagram (for three sets)



# Inclusion-Exclusion (for three sets)

Generic form:

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad -(|A \cap B| + |A \cap C| + |B \cap C|) \\ &\quad +|A \cap B \cap C| \end{aligned}$$

Alternative form:

$$\begin{aligned} |A \cap B \cap C| &= |X| \\ &\quad -(|A^c| + |B^c| + |C^c|) \\ &\quad +(|A^c \cap B^c| + |A^c \cap C^c| + |B^c \cap C^c|) \\ &\quad -|A^c \cap B^c \cap C^c| \end{aligned}$$

Can you see a pattern?

## Inclusion-Exclusion (for $n$ sets)

Given $S_1, \ldots, S_n \subseteq X$.
Generic form:

$$|S_1 \cup \cdots \cup S_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |S_{i_1} \cap \cdots \cap S_{i_k}|$$

Alternative form:

$$|S_1 \cap \cdots \cap S_n| = \sum_{k=0}^{n} (-1)^{k} \sum_{1 \le i_1 < \cdots < i_k \le n} |S_{i_1}^c \cap \cdots \cap S_{i_k}^c|$$

Implementation is left as exercise. (Two common ways: do it recursively, or use bitwise operations.)

# Bit String Representation of Subsets

Given $S_1, \ldots, S_n \subseteq X$, we can partition $X$ into $2^n$ disjoint subsets, labelled with binary strings of length $n$.

$t = t_1 t_2 \ldots t_n$ labels the subset containing elements:

$$\begin{cases} \text{in } S_i, & \text{if } t_i = 1 \\ \text{in } S_i^c \text{ (i.e. not in } S_i\text{)}, & \text{if } t_i = 0 \end{cases}$$

For example, $n = 2$:

- Set $00$ is $S_1^c \cap S_2^c$
- Set $01$ is $S_1^c \cap S_2$
- Set $10$ is $S_1 \cap S_2^c$
- Set $11$ is $S_1 \cap S_2$

Call them **atomic sets**.

## Using $?$

It is convenient to consider sets like $0?0?$, $???1$, and $????$, which contain atomic sets with matching string patterns.

Example 1
$0?0?$ contains four atomic sets: $0000, 0001, 0100, 0101$.
$0?0?$ can be translated as: $S_1^c \cap S_3^c$, i.e. elements not contained in $S_1$ or $S_3$.

Example 2
$???1$ contains eight atomic sets: $0001, 0011, 0101, 0111, 1001, 1011, 1101, 1111$.
$???1$ can be translated as: $S_4$, i.e. elements contained in $S_4$.

Example 3
$????$ is just the whole set $X$.

## Proving Inclusion-Exclusion Formula

$$|S_1 \cup \cdots \cup S_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |S_{i_1} \cap \cdots \cap S_{i_k}|$$

Idea: For each atomic set, check that coefficients are the same on both sides.

- For $00...0$, coefficients on both sides are $0$.
- For atomic set with $r > 0$ $1$'s:
  - Coefficient on LHS is $1$.
  - Coefficient on RHS is $\sum_{k=1}^{n} (-1)^{k-1} \binom{r}{k} = 1$. (Think about: how many patterns with $k$ $1$'s and $(n-k)$ ?'s match the bit string with $r$ $1$'s? It is $\binom{r}{k}$.)

## Alternative Form

$$|S_1 \cap \cdots \cap S_n| = \sum_{k=0}^{n} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} |S_{i_1}^c \cap \cdots \cap S_{i_k}^c|$$

- For $11...1$, coefficients on both sides are $1$.
- For atomic set with $r > 0$ $0$'s:
  - Coefficient on LHS is $0$.
  - Coefficient on RHS is $\sum_{k=0}^{n} (-1)^k \binom{r}{k} = 0$. (Think about: how many patterns with $k$ $0$'s and $(n-k)$ $?$'s match the bit string with $r$ $0$'s? It is $\binom{r}{k}$.)

## Solving M1832

Problem statement: find the number of integral solutions to $x_1 + \cdots + x_n = T$, subject to $0 \le x_i \le a_i$. Output the answer modulo $10^9 + 7$.

Constraints: $1 \le n \le 16, 1 \le T \le 10^9, 1 \le a_i \le 10^9$.

# Solving M1832

Problem statement: find the number of integral solutions to $x_1 + \cdots + x_n = T$, subject to $0 \le x_i \le a_i$. Output the answer modulo $10^9 + 7$.

Constraints: $1 \le n \le 16, 1 \le T \le 10^9, 1 \le a_i \le 10^9$.

Solution Idea:

- Let $X$ be the solution set to $x_1 + \cdots + x_n = S$, without upper bound constraints.
- Let $S_i$ consist of elements of $X$ with $x_i \le a_i$.
- We want to find $|S_1 \cap S_2 \cap \cdots \cap S_n|$.
- Handling constraint violation is easy! If $x_i > a_i$ we just use $x_i' := x_i - a_i - 1$ to get back an unconstrained equation. So we know how to calculate quantities like $|S_{i_1}^c \cap \cdots \cap S_{i_k}^c|$ quickly.
- Use Inclusion-Exclusion (alternative form) to find the answer.

# Inclusion-Exclusion With A Twist (M1933)

Problem statement: given a graph with $D \leq 20$ edges, find the number of subgraphs with **exactly** one edge. Output the answer modulo $10^9 + 7$.

Think in terms of bit strings and atomic sets.

# Inclusion-Exclusion With A Twist (M1933)

Problem statement: given a graph with $D \leq 20$ edges, find the number of subgraphs with **exactly** one edge. Output the answer modulo $10^9 + 7$.

Think in terms of bit strings and atomic sets.

- Let $X$ be the set of all subgraphs.
- Let $S_i$ be the set of subgraphs containing edge $i$.
- Building block: strings with $1$'s and $?$'s (we cannot handle $0$).
- Target: $10..0$, $010..0$, ..., $00..01$ have coefficients $1$, the rest $0$.

# M1933 Continued

Target: $10..0$, $010..0$, ..., $00..01$ have coefficients $1$, the rest $0$.
For simplicity consider $D = 3$.
From Inclusion-Exclusion (alternative form), we know:

$$00 = ?? - 1? - ?1 + 11$$

Inserting $1$ in three different positions, we get:

$$\begin{aligned} 100 &= 1?? - 11? - 1?1 + 111 \\ 010 &= ?1? - 11? - ?11 + 111 \\ 001 &= ??1 - 1?1 - ?11 + 111 \end{aligned}$$

Translating back to set notation, the desired equation is

$$|S_1| + |S_2| + |S_3| - 2 \times (|S_1 \cap S_2| + |S_1 \cap S_3| + |S_2 \cap S_3|) + 3 \times |S_1 \cap S_2 \cap S_3|$$

## M1933 Continued

In fact the general formula is

$$\sum_{k=1}^{n} k \times (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |S_{i_1} \cap \cdots \cap S_{i_k}|.$$

Think about: How to calculate the number of subgraphs containing exactly $E$ ($0 \le E \le D$) edges? (Surprise, surprise... binomial coefficients will emerge!)

# Table of Contents

# Introduction to Probability Space

A (finite) probability space consists of two parts:

1. A set $\Omega = \{a_1, \ldots, a_n\}$ of outcomes
2. Associated probabilities $p_1, \ldots, p_n$ of each outcome ($\sum_i p_i = 1$)

Probabilities can be abstract (based on idealised model) or empirical (based on statistical evidence).
Usually we consider uniform probability spaces, i.e. $p_1 = \cdots = p_n = \frac{1}{n}$.

Examples:

- Dice roll
- Coin toss
- A sequence of $N$ coin tosses
- Drawing a ball from a bag of colored balls

## Event Probability

An event is (a short description of) a subset of $\Omega$.
For example:

- $\Omega_1 :=$ sequences of $3$ coin tosses, $A_1 :=$ seqs with two consecutive heads.
  Then, $A_1 = \{HHH, HHT, THH\}$.
  $P(A_1) = \frac{3}{8}$.
- $\Omega_2 :=$ dice roll, $A_2 :=$ odd outcomes.
  Then, $A_2 = \{1, 3, 5\}$.
  $P(A_2) = \frac{3}{6} = \frac{1}{2}$.

For uniform probability spaces,

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of possible outcomes}},$$

so we just need to count :)

## Conditional Probability

For two events $A$ and $B$ with $P(B) > 0$, **define** conditional probability as

$$P(A|B) := \frac{P(A \cap B)}{P(B)}.$$

For DP on probabilities, we often use the fact that

$$\underbrace{P(A \cap B)}_{\text{new state}} = \underbrace{P(A|B)}_{\text{transition}} \underbrace{P(B)}_{\text{previous state}}$$

Exercise: Suppose we pick an integer $x$ in $[1, 20]$ uniformly at random.
Calculate $P(x$ is odd | $x$ is prime$)$ and $P(x$ is prime | $x$ is odd$)$.

# Random Variable, Expectation, Variance

A random variable is a function $f : \Omega \longrightarrow \mathbb{R}$.
It assigns a value of each of $a_1, \ldots, a_n$.

Examples: slot machine, dice roll

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**More Definitions**

Expectation of $f$ : $\mathbb{E}[f] := \sum_{i=1}^{n} f(a_i) p_i.$

Variance of $f$ : $Var(f) := \mathbb{E}[(f - \mathbb{E}[f])^2].$

$\mathbb{E}[f]$ is the average value of $f$, weighted by probability (think lottery).
$Var(f)$ measures how $f$ deviates from its average.

## A Remark

Event probability is just the expected value of certain **indicator functions**.
Just choose $f(x) := 1$ if $x \in A$ and $f(x) := 0$ otherwise.
Then $\mathbb{E}[f] = P(A)$.
We denote this function by $\mathbb{1}_A$.

# Linearity of Expectation

**Theorem (Linearity of Expectation)**

Suppose random variable $f$ is written as sum $f_1 + \cdots + f_k$. Then,

$$\mathbb{E}[f] = \sum_{i=1}^{k} \mathbb{E}[f_i].$$

(Think about: do we have "linearity of variance"?) NO.

This simple theorem is very powerful!

Core Idea: Break down a quantity into small, easy-to-calculate parts.

# Example 1

Suppose we pick a permutation $p = [p_1, \ldots, p_n]$ of $1..n$ uniformly at random. What is the expected number of fixed points of $p$?

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Suppose we pick a permutation $p = [p_1, \ldots, p_n]$ of $1..n$ uniformly at random. What is the expected number of fixed points of $p$?

Let $X$ be the number of fixed points of $p$. Then $X = X_1 + \cdots + X_n$ where $X_i$ is indicator function for $p_i = i$.
By linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \textstyle\sum_{i=1}^{n} \mathbb{E}[X_i] \\ &= \textstyle\sum_{i=1}^{n} P(p_i = i) \\ &= \textstyle\sum_{i=1}^{n} \frac{1}{n} \\ &= 1.\end{aligned}$$

## Example 2

$N > 2$ people form a circle. Each person gives a coin to the person to his left or to his right, each with probability $\frac{1}{2}$.
What is the expected number of people who receives no coins?

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What is the expected number of people who receives no coins?

Let $X$ be the number of people who receives no coins. Then $X = X_1 + \cdots + X_N$ where $X_i$ is indicator function for "person $i$ receives no coins".
By linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1}^{n} \mathbb{E}[X_i] \\ &= \sum_{i=1}^{n} P(\text{person } i \text{ receives no coins}) \\ &= \sum_{i=1}^{n} \frac{1}{4} \\ &= \frac{n}{4}.\end{aligned}$$

## Example 3 (c.f. CF453A)

You roll a $6$-sided die $N$ times.
What is the expected value of the biggest number obtained?

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You roll a $6$-sided die $N$ times.
What is the expected value of the biggest number obtained?

Let $X$ be the biggest number obtained. Then $X = X_1 + \cdots + X_6$ where $X_i$ is indicator function for "exists a number $\geq i$".
By linearity of expectation,

$$\begin{aligned}
\mathbb{E}[X] &= \textstyle\sum_{i=1}^{6} \mathbb{E}[X_i] \\
&= \textstyle\sum_{i=1}^{6} P(\text{exists a number } \geq i) \\
&= \textstyle\sum_{i=1}^{6} (1 - P(\text{all numbers smaller than } i)) \\
&= \textstyle\sum_{i=1}^{6} (1 - (\frac{i-1}{6})^n) \\
&= 6 - \frac{1^n+2^n+3^n+4^n+5^n}{6^n}
\end{aligned}$$

## Example 4

Generate a sequence of length $n$, where each element is chosen independently and uniformly from $1..k$.
What is the expected number of distinct elements?

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Let $X$ be the number of distinct elements. Then $X = X_1 + \cdots + X_k$ where $X_i$ is indicator function for "number $i$ appears in the sequence".
By linearity of expectation,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{i=1}^{k} \mathbb{E}[X_i] \\
&= \sum_{i=1}^{k} P(\text{number } i \text{ appears in the sequence}) \\
&= \sum_{i=1}^{k} (1 - P(\text{number } i \text{ does not appear in the sequence})) \\
&= \sum_{i=1}^{k} (1 - (\tfrac{k-1}{k})^n) \\
&= k(1 - (\tfrac{k-1}{k})^n)
\end{aligned}$$

## Example 5 (CF 280C)

Given a rooted tree with $n$ nodes. While there are $k > 0$ nodes remaining, pick one node $Q$ uniformly at random and remove the subtree rooted at $Q$.
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What is the expected number of operations?

Let $X$ be the number of operations. Then $X = X_1 + \cdots + X_n$ where $X_i$ is indicator function for "node $i$ is picked".
The event that node $i$ is picked only depends on node $i$ and its $H[i]$ ancestors.
To eliminate node $i$, exactly one of the $(H[i] + 1)$ nodes needs to be picked, each with equal probability.
Therefore, expected number of operations is $\sum_{i=1}^{n} \frac{1}{H[i]+1}$.

## Example 6

Given a convex polygon with $N$ vertices. Choose a random subset of vertices, each vertex chosen with probability $\frac{1}{2}$ and independent of other vertices. Form a new polygon by connecting adjacent chosen vertices. What is the expected perimeter of the new polygon?

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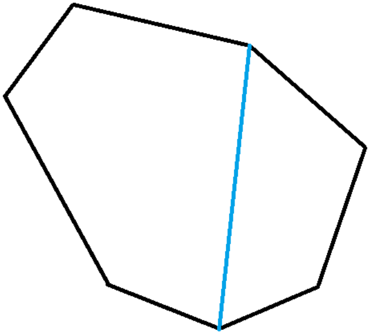
A diagonal is a part of the perimeter if and only if:

- Its two endpoints are chosen
- All points on at least one side are not chosen

Notice that if the endpoints are too far away (in terms of number of vertices in between), the probability is negligible.



# Expectation and Counting

## Same Core Idea:
## Break down a quantity into small, easy-to-calculate parts.

Question on expectation:
Suppose we pick a permutation $p = [p_1, \ldots, p_n]$ of $1..n$ uniformly at random.
What is the **expected number** of fixed points of $p$?

Can be easily translated into

Question on counting:
Consider all permutations $p = [p_1, \ldots, p_n]$ of $1..n$. What is the **total number** of fixed points?

## Counting Fixed Points – $n = 4$

| p[1] | p[2] | p[3] | p[4] | fixed pts | | p[1] | p[2] | p[3] | p[4] | fixed pts |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 4 | | 3 | 1 | 2 | 4 | 1 |
| 1 | 2 | 4 | 3 | 2 | | 3 | 1 | 4 | 2 | 0 |
| 1 | 3 | 2 | 4 | 2 | | 3 | 2 | 1 | 4 | 2 |
| 1 | 3 | 4 | 2 | 1 | | 3 | 2 | 4 | 1 | 1 |
| 1 | 4 | 2 | 3 | 1 | | 3 | 4 | 1 | 2 | 0 |
| 1 | 4 | 3 | 2 | 2 | | 3 | 4 | 2 | 1 | 0 |
| 2 | 1 | 3 | 4 | 2 | | 4 | 1 | 2 | 3 | 0 |
| 2 | 1 | 4 | 3 | 0 | | 4 | 1 | 3 | 2 | 1 |
| 2 | 3 | 1 | 4 | 1 | | 4 | 2 | 1 | 3 | 1 |
| 2 | 3 | 4 | 1 | 0 | | 4 | 2 | 3 | 1 | 2 |
| 2 | 4 | 1 | 3 | 0 | | 4 | 3 | 1 | 2 | 0 |
| 2 | 4 | 3 | 1 | 1 | | 4 | 3 | 2 | 1 | 0 |
| | | | | | SUM | 6 | 6 | 6 | 6 | 24 |

## Example 7

Given an integer set $S = \{b_1, b_2, \dots, b_n\}$. For each subset $T$ of $S$, let $sum(T)$ denote the sum of elements of $T$. Calculate

$$\sum_{T \subseteq S} sum(T) \mod M.$$

## Example 7

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$$\sum_{T \subseteq S} sum(T) \mod M.$$

Let

$$contain_i(T) := \begin{cases} 1, & \text{if } T \text{ contains } b_i; \\ 0, & \text{otherwise.} \end{cases}$$

Then $sum(T) = \sum_{i=1}^{n} b_i \times contain_i(T)$, and

$$\begin{aligned} \sum_{T \subseteq S} sum(T) &= \sum_{T \subseteq S} \sum_{i=1}^{n} b_i \times contain_i(T) \\ &= \sum_{i=1}^{n} b_i \left( \sum_{T \subseteq S} contain_i(T) \right) \\ &= \sum_{i=1}^{n} b_i 2^{n-1} \end{aligned}$$

# Example 8

Given an array $a[1..n]$. For $1 \le l \le r \le n$, let $sum(l, r)$ denote $a[l] + a[l+1] + \cdots + a[r]$. Calculate

$$\sum_{1 \le l \le r \le n} sum(l, r).$$

# Example 8

Given an array $a[1..n]$. For $1 \le l \le r \le n$, let $sum(l,r)$ denote $a[l] + a[l+1] + \cdots + a[r]$. Calculate

$$\sum_{1 \le l \le r \le n} sum(l,r).$$

Let

$$contain_p(l,r) := \begin{cases} 1, & \text{if } l \le p \le r; \\ 0, & \text{otherwise.} \end{cases}$$

Then $sum(l,r) = \sum_{p=1}^{n} a[p] \times contain_p(l,r)$, and

$$\begin{aligned} \textstyle\sum_{1 \le l \le r \le n} sum(l,r) &= \textstyle\sum_{1 \le l \le r \le n} \sum_{p=1}^{n} a[p] \times contain_p(l,r) \\ &= \textstyle\sum_{p=1}^{n} a[p] \times p \times (n+1-p) \end{aligned}$$

# Example 9

Given an integer set $S = \{s_1, s_2, \ldots, s_n\}$. For each nonempty subset $T$ of $S$, let $diff(T)$ denote the difference between the largest and the smallest elements of $T$. Calculate

$$\sum_{\emptyset \neq T \subseteq S} diff(T) \mod M.$$

# Example 9

Given an integer set $S = \{s_1, s_2, \ldots, s_n\}$. For each nonempty subset $T$ of $S$, let $diff(T)$ denote the difference between the largest and the smallest elements of $T$. Calculate

$$\sum_{\varnothing \neq T \subseteq S} diff(T) \mod M.$$

Sort the numbers. For $1 \leq p < n$, let

$$contain_p(T) := \begin{cases} 1, & \text{if } min\_index(T) \leq p < max\_index(T) \\ 0, & \text{otherwise.} \end{cases}$$

Then $diff(T) = \sum_{p=1}^{n-1} (s_{p+1} - s_p) \times contain_p(T)$, and

$$\begin{aligned} \sum_{\varnothing \neq T \subseteq S} diff(T) &= \sum_{\varnothing \neq T \subseteq S} \sum_{p=1}^{n-1} (s_{p+1} - s_p) \times contain_p(T) \\ &= \sum_{p=1}^{n-1} (s_{p+1} - s_p) \times \sum_{\varnothing \neq T \subseteq S} contain_p(T) \\ &= \sum_{p=1}^{n-1} (s_{p+1} - s_p) \times (2^p - 1) \times (2^{n-p} - 1) \end{aligned}$$

# Example 10

Given a tree with $N$ nodes. Let $l(x, y)$ be the length (number of edges) of the simple path from $x$ to $y$. Calculate

$$\sum_{x,y} l(x, y).$$

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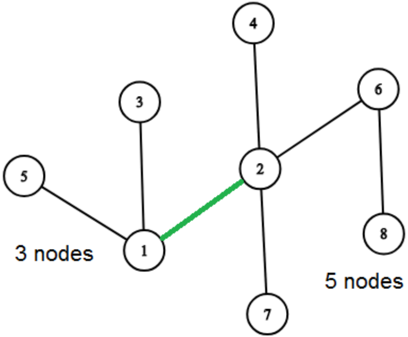
Given a tree with $N$ nodes. Let $l(x,y)$ be the length (number of edges) of the simple path from $x$ to $y$. Calculate

$$\sum_{x,y} l(x,y).$$

For each edge, count the number of paths that pass through it.
It is (number of nodes on one side) $\times$ (number of nodes on the other side).
This can be done using a DFS on tree.



## Example 11

Given a complete graph. Each edge is colored red or blue. Count the number of monochromatic triangles. Do better than $O(|V|^3)$ :)

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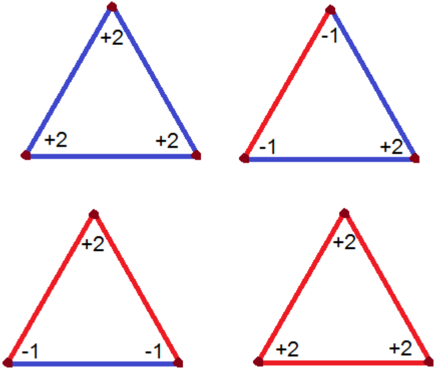
Hint:



# Further Reading

Proofs That Really Count (The Art of Combinatorial Proof)
by Arthur T. Benjamin and Jennifer J. Quinn

Sums and Expected Value – Part 1 (https://codeforces.com/blog/entry/62690)
Sums and Expected Value – Part 2 (https://codeforces.com/blog/entry/62792)
by Errichto