Appendix: Why $C[i][j] \leq C[i][j+1]$ in CF321E

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In today's DP (III) lecture, I promised to write a proof that D&C optimization can be used for solving Ciel and Gondolas. Here it is.

Recall that $dp[i][j]$:= minimal cost of partitioning $[1, j]$ into i groups, and we have the simple transition formula

$$dp[i][j] = min_{k<j}(dp[i-1][k] + f(k+1, j))$$

where $f(l, r)$ equals the sum of elements in the subarray $s[l..r][l..r]$.

1 The First Step

To simplify notations a bit, assume we are transitioning from $old[0..n]$ to $dp[0..n]$ (instead of from $dp[i-1][0..n]$ to $dp[i][0..n]$, and write $C[j]$ for $C[i][j]$. (All this is done to get rid of the somewhat irrelevant index i.)

So we want to prove $C[j] \leq C[j+1]$, where $C[j]$ is the smallest index k minimizing $old[k] + f(k+1, j)$. It is usually a good idea to prove *by contradiction*: suppose $C[j] > C[j+1]$, then there must be something wrong.

2 Inequalities

Here are two simple inequalities.

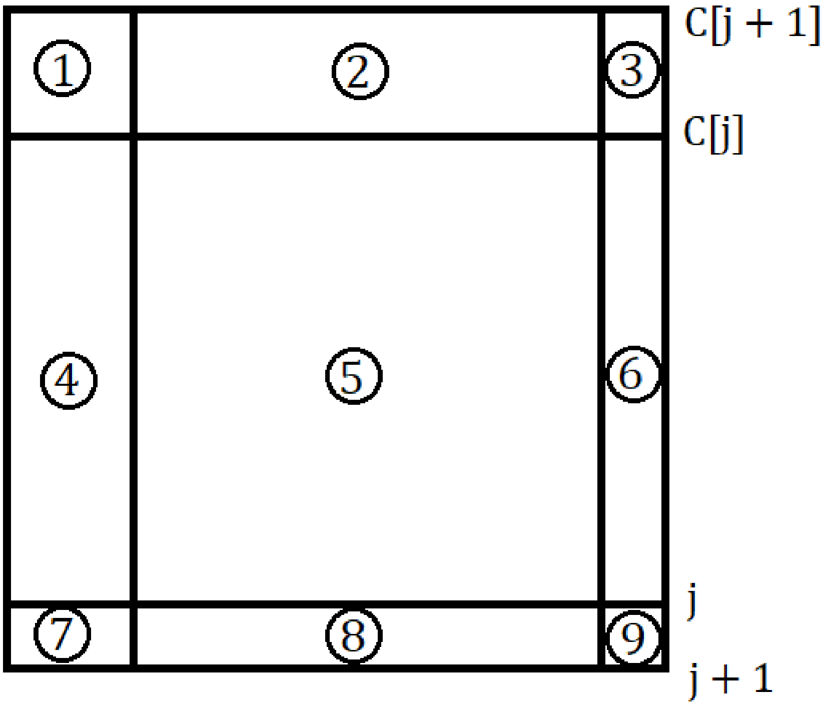
- $old[C[j]] + f(C[j]+1, j) < old[C[j+1]] + f(C[j+1]+1, j)$
  ($\because C[j]$ is the best index for $dp[j]$.)
- $old[C[j+1]] + f(C[j+1]+1, j+1) \leq old[C[j]] + f(C[j]+1, j+1)$
  ($\because C[j+1]$ is the best index for $dp[j+1]$.)

Add them up to get

$$f(C[j]+1, j) + f(C[j+1]+1, j+1) < f(C[j]+1, j+1) + f(C[j+1]+1, j).$$

3 Why is this absurd?

This is (almost) a proof without words:



So our assumption $C[j] > C[j+1]$ has led us to the inequality $f(C[j]+1, j) + f(C[j+1]+1, j+1) < f(C[j]+1, j+1) + f(C[j+1]+1, j)$. But...

$$LHS = \textcircled{5} + (\textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4} + \textcircled{5} + \textcircled{6} + \textcircled{7} + \textcircled{8} + \textcircled{9})$$

$$RHS = (\textcircled{1} + \textcircled{2} + \textcircled{4} + \textcircled{5}) + (\textcircled{5} + \textcircled{6} + \textcircled{8} + \textcircled{9})$$

$$\therefore LHS - RHS = \textcircled{3} + \textcircled{7} \geq 0. \text{ Contradiction!}$$