Advanced Divide & Conquer

HKOI Training 9-3-2019

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Agenda

- Basics of Divide & Conquer
- Divide & Conquer on Range Query Problem
- Divide & Conquer on Tree (Centroid Decomposition)
- Divide & Conquer on Contribution Technique (CDQ Divide & Conquer)

Basics of D&C

Basics of D&C

- We have a problem f with parameter n, f(n)
- We can divide it to a SAME problem with SMALLER parameter f(m) ($m < n$)
- By solving the f(m) first, we can solve the original problem much easier!
  - f(m) may help us to compute f(n) easily
  - Or by excluding f(m) from f(n), we can reduce f(n) to an easier problem g(n)

Basics of D&C – Sum of Geometric Sequence

- Given a, k, m (m may not be a prime)
- Find (a^0 + a^1 + a^2 + … a^k) % m

- Solution 1:
- General Formula: a^0 + a^1 + … + a^k = (a^k - 1) / (a - 1)

Basics of D&C – Sum of Geometric Sequence

- Solution 1:
- General Formula: a^0 + a^1 + … + a^k = (a^k - 1) / (a - 1)
- However, if gcd(a – 1, m) != 1, we may not able to find the modular inverse
- ☹

Basics of D&C – Sum of Geometric Sequence

- Solution 2:
- Let $f(n) = (a^0 + a^1 + \ldots + a^n) \% m$
- What if we know the answer of $f(n / 2)$?
- $f(n / 2) = (a^0 + a^1 + \ldots a^{(n/2)}) \% m$

- Does the answer of $f(n / 2)$ able to help us find $f(n)$ easily?

Basics of D&C – Sum of Geometric Sequence

- When n is odd
- $f\left(\frac{n}{2}\right) \times (a^{1+\frac{n}{2}}+1) = \left(a^0 + a^1 + \cdots + a^{\frac{n}{2}}\right)\left(a^{\frac{n}{2}+1} + 1\right)$
  $= a^0 + a^1 + \cdots + a^{\frac{n}{2}} + a^{\frac{n}{2}+1} + \cdots + a^n$
- $= f(n)$
- Similarly, when n is even
- $f\left(\frac{n}{2}\right) \times (a^{\frac{n}{2}}+1) - a^{\frac{n}{2}} = (a^0 + a^1 + \cdots + a^n)\ as\ well$

Basics of D&C – Sum of Geometric Sequence

- So, if we have known the value of f(n / 2)
- We just need to know
  - a^(1 + n/2) + 1 to find f(n) in odd case
  - a^(n/2) + 1 and a^(n/2) to find f(n) in even case
- Where a^k can be found by a BigMod algorithm
- ☺

Basics of D&C – Sum of Geometric Sequence

- Time complexity:
- To calculate f(n), we need the value of f(n / 2)
  - → we need to calculate log(n) value of f()
- Calculating f(n) by f(n / 2) require us to find a^(n/2) e.t.c.
- i.e. we need to do BigMod for log(n) times
- Time complexity: O((log n)^2)   (Actually O(log n) with careful analysis)

Basics of D&C

- We have a problem f with parameter n, f(n)
- We can divide it to a SAME problem with SMALLER parameter f(m) ($m < n$)
- By solving the f(m) first, we can solve the original problem much easier!
  - f(m) may help us to compute f(n) easily → the above example
  - Or by excluding f(m) from f(n), we can reduce f(n) to an easier problem g(n)
- The following more advanced examples are about the 2nd type reduction

D&C on Range Query

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D&C on Range Query

- Given an Array A[1..n] and Q query (offline)
- l, r is given in each query
- For each query, find gcd(A[l], A[l + 1], … A[r])

D&C on Range Query

- Firstly, you may have seen a similar problem to find sum(A[l], A[l + 1] … A[r]) instead of gcd(A[l], A[l + 1] … A[r])
- You can use partial sum to solve the sum version because:
  - Sum(l, r) = Sum(1, r) - Sum(1, l - 1)
- However, in gcd version, minus (-) operator is undefined
- We can only define the add operator for gcd version

D&C on Range Query

- Is it possible to extend the partial sum idea when minus operation is not defined?

- YES!!! With the help of divide & conquer

D&C on Range Query

- Consider a easier version of the original problem first:
  - For each query (l, r), $l <= n/2 <= r$
- In this case, we can compute two partial gcd array
  - $gcdA[i] = gcd(A[i], A[i + 1] \ldots A[n/2])$ for all $i <= n/2$
  - $gcdB[i] = gcd(A[n/2], A[n/2 + 1] \ldots A[i])$ for all $i >= n / 2$
- To get the answer of query(l, r) where $l <= n/2 <= r$:
  - $Res = gcd(gcdA[l], gcdB[r])$

D&C on Range Query

- E.g. A = {2, 4, 6, 12, 3, 9, 6, 7} → n = 8, n / 2 = 4
- gcdA = {2, 2, 6, 12} for 1 <= i <= 4
- gcdB = {12, 3, 3, 3, 1} for 4 <= i <= 8

- E.g. we want to find gcd(3, 6) → gcd(gcdA[3], gcdB[6]) = gcd(6, 3) = 3
- Solve in O(log n)! (just find gcd of two number)

D&C on Range Query

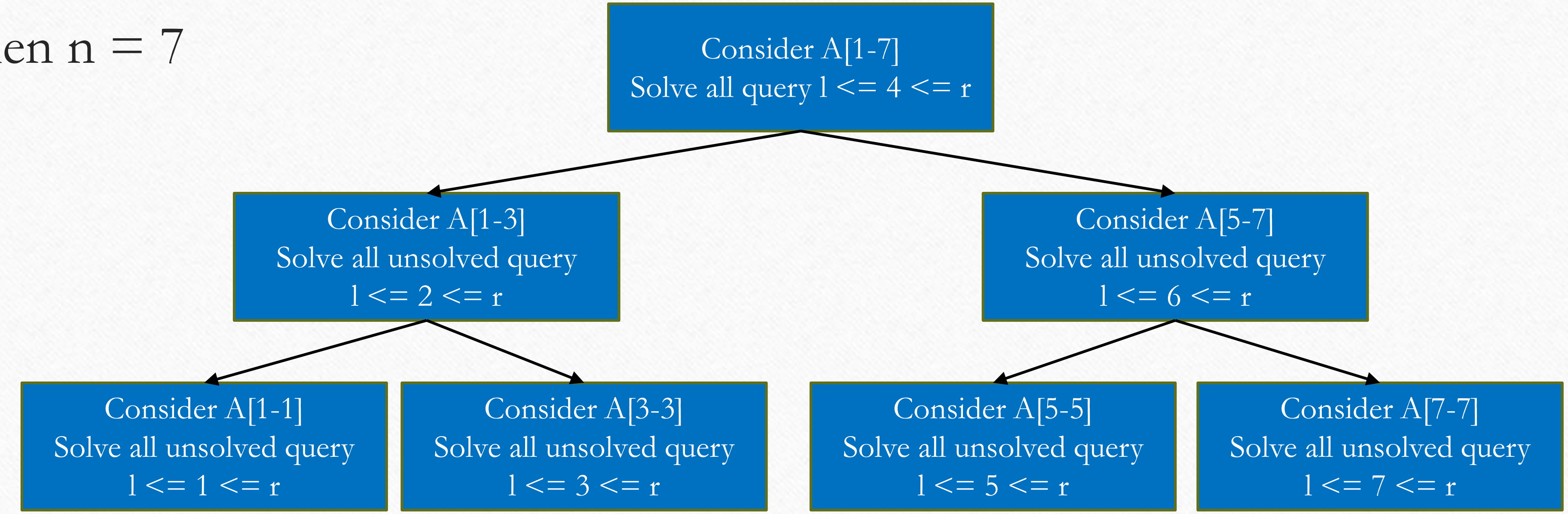
- To get the answer of query(l, r) where l <= n/2 <= r:
  - Res = gcd(gcdA[l], gcdB[r])
- We overcome the minus operator by building the partial gcd array from n/2
- However what if the query(l, r) do not satisfy l <= mid <= r?

D&C on Range Query

- However what if the query(l, r) do not satisfy l <= mid <= r?
- Divide & Conquer help!
- After solving all case with l <= mid <= r
- We just care about the cases where:
  - l <= r < mid → Consider the first half of array A only
  - mid < l <= r → Consider the second half of array A only
  - Which is the same problem with smaller scale

D&C on Range Query

- When n = 7



D&C on Range Query

- When n = 7,
  query: [1, 4], [3, 5], [4, 6], [5, 6], [7, 7], [1, 3]
- For the 1st instance, consider A[1-7] → solve all query l <= 4 <= r
  - [1, 4], [3, 5], [4, 6]
- For the 2nd instance, consider A[1-3] → solve all unsolved query l <= 2 <= r
  - [1, 3]
- …

D&C on Range Query

- Time complexity for one instance to compute the partial gcd array:
  - O(n + log M) where M is the largest value
- Time complexity for all instance to compute the partial gcd array:
  - O(n log n + n log M)
- Time complexity to answer all the query: O(Q log M)
- Total time complexity: O((n + Q) * log(n + M)) → one log only

D&C on Range Query

- Somebody may think of using segment tree to solve Range Query problem
- It is usually Okay but sometimes D&C can give a faster time complexity!

D&C on Range Query

- Problem:
- Given array A[1..n] where A[i] < 20
- Q query l, r
- For each query, find number of subsequence in subarray A[l, r] such that sum of subsequence % 20 == 0

D&C on Range Query

- Solution D&C + dp or segment tree + dp
- In D&C, we use partial sum concept to store a partial dp value
  - dpA[i][k] = number of way to use A[i] to A[mid] to make a subset sum = k (mod 20)
  - dpB[i][k] = number of way to use A[mid+1] to A[i] to make a subset sum = k (mod 20)
- However, segment tree time complexity will be O(Q log n * 20^2)
- D&C will be O(n log n * 20 + 20 * Q), faster !!!

D&C on Range Query

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- However, segment tree time complexity will be O(Q log n * 20^2)
- D&C will be O(n log n * 20 + 20 * Q), faster !!!

D&C on Range Query

- In short: Steps to use D&C to solve range query problem
  - Think whether it can be solved easily for query l <= mid <= r
  - Put the queries to the suitable instance to solve it
  - Use recursion to code the D&C part!

D&C on Range Query

- Let’s code together:
- M0921 (Range maximum query)
- https://codeforces.com/gym/101741/problem/J (Range Subsequence sum)

D&C on Tree

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Common Form for Tree Query Problem

- If you encounter an tree problem asking:
  - Count total number of path satisfying xxxxxx
  - Consider all the path, find the optimal pathing satisfying xxxxxx
- Then, the problem is usually able to be solved by D&C on Tree

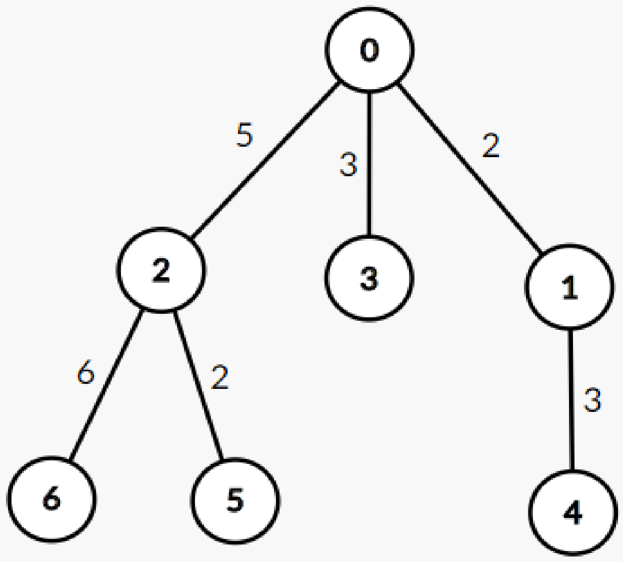
IOI 2011 Race

- Given a weighted unrooted tree
- Find number of pair(x, y)
    - satisfying distance between node x and node y = K where K is a constant

IOI 2011 Race

- Assume K = 8
- The answer = 3
- {(2, 3), (3, 4), (5, 6)}

- An O(N^2) solution can be achieved easily by N DFS

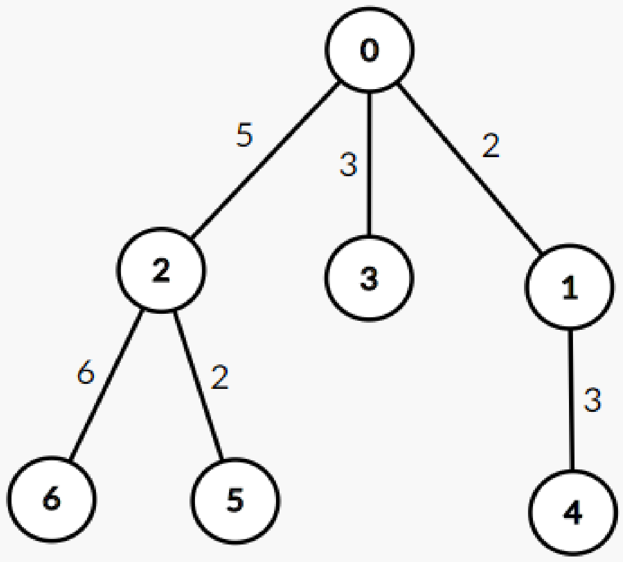


IOI 2011 Race

- To achieve a better solution, we can……
- Consider an easier version first
- find number of pair(x, y)
  - satisfying distance between node x and node y = K where K is a constant
  - and the path between x and y must pass through node 0

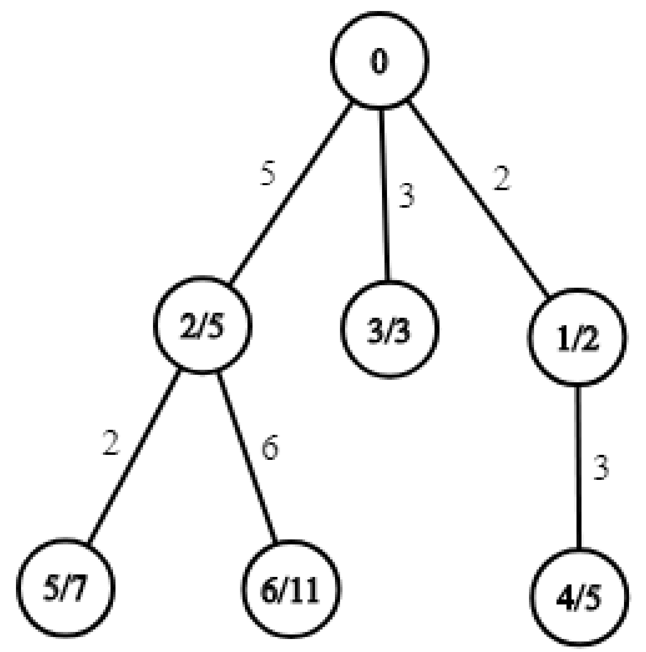
IOI 2011 Race (easier version)

- Assume K = 8
- The answer = 2
- {(2, 3), (3, 4)}
- (2 → 0 → 3), (3 → 0 → 1 → 4)



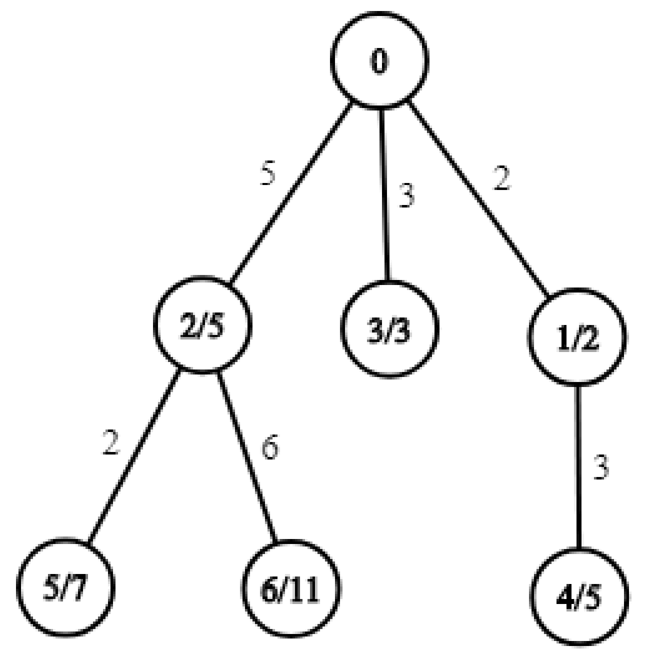
IOI 2011 Race (easier version)

- Let's fix node 0 as root
- Compute the distance from 0 to every node
  - Let's denote as dist[u]
- Then, for a pair of node (u, v), if
  - dist[u] + dist[v] == k
  - path(u, v) passing through 0
- Then path(u, v) satisfy the constraints



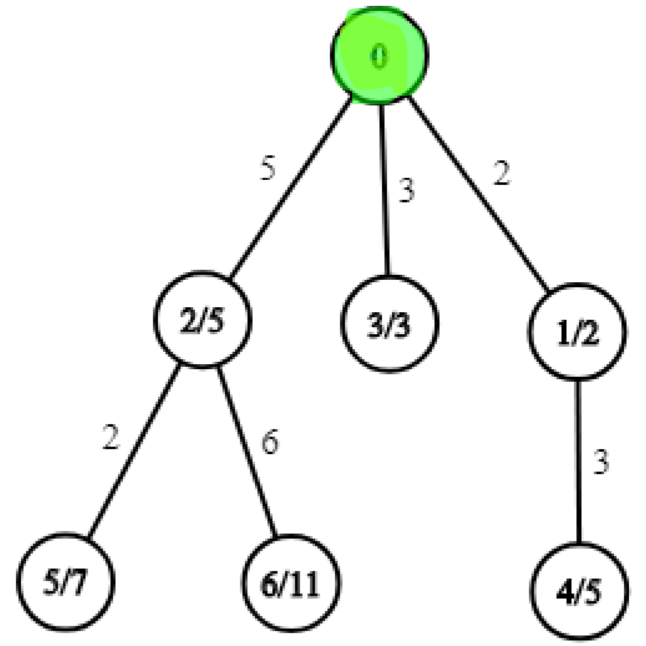
IOI 2011 Race (easier version)

- To find all pairs satisfying dist[u] + dist[v] = k:
  - When iterate each node u by DFS order from 0
  - ans += freq[k - dist[u]];
  - freq[dist[u]] += 1;
- To ensure it pass through node 0
  - When iterate each node u by DFS order from 0
  - ans += freq[k – dist[u]]
  - But only update freq[] when we finish iterating a whole subtree of 0



IOI 2011 Race (easier version)

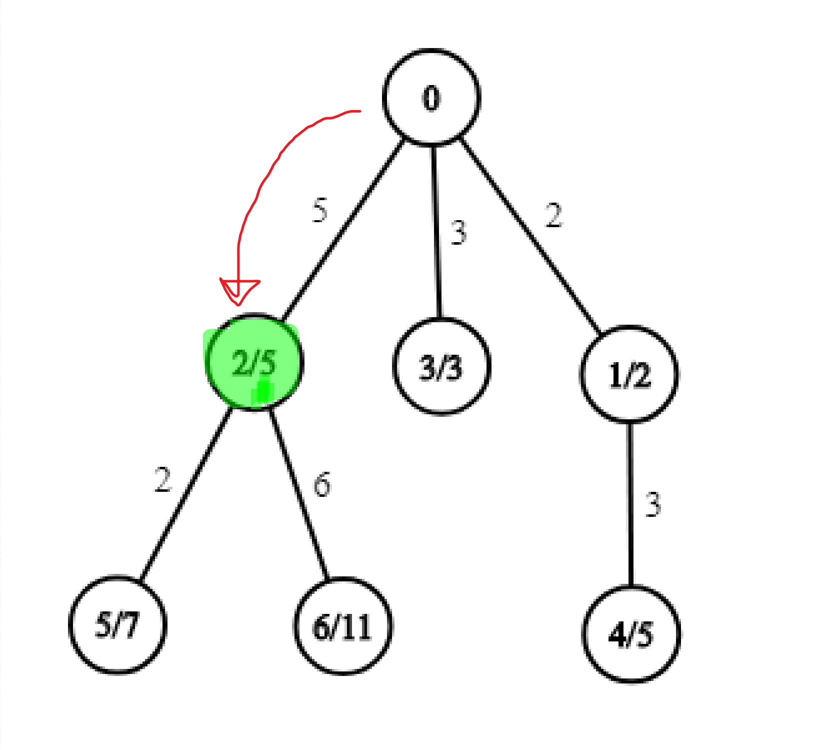
- We start iterating at node 0
- `Ans += freq[k - 0]`
- `Freq[0]++;`



IOI 2011 Race (easier version)

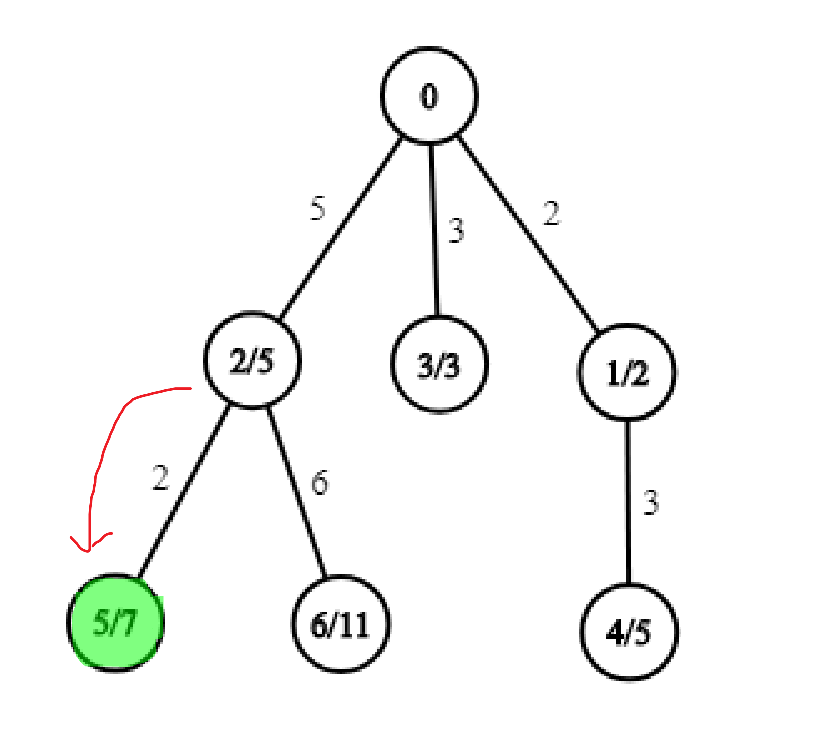
- `Ans += freq[k - 5]`

- Note that we ***won't*** perform freq[5]++;
- As we haven't iterate all the node in this subtree {2, 5, 6}
- To avoid counting path that not passing 0, we should not freq[5]++ currently



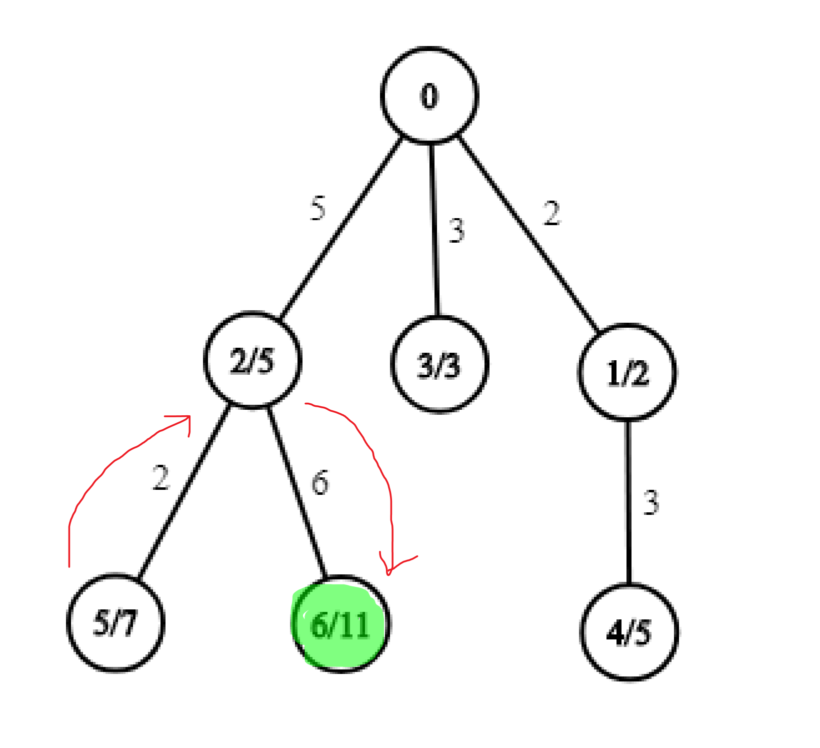
IOI 2011 Race (easier version)

- Ans += freq[k – 7]



IOI 2011 Race (easier version)

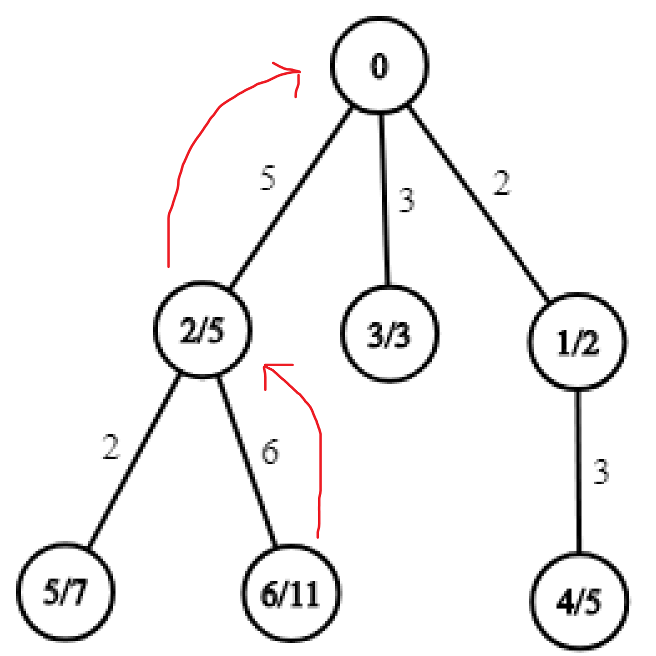
- `Ans += freq[k – 11]`



IOI 2011 Race (easier version)

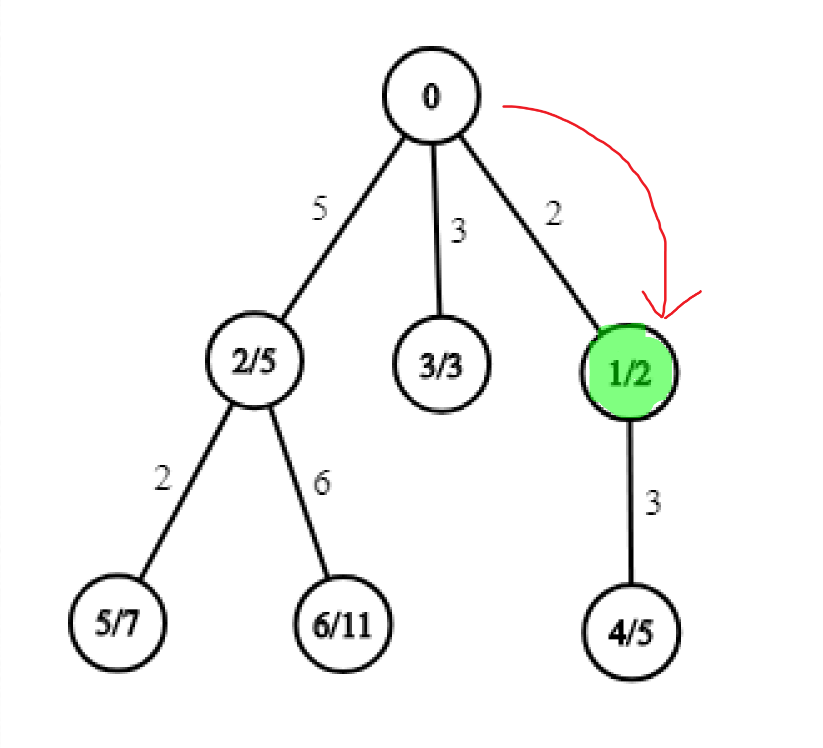
- Note that when our DFS go back to node 0
- This means we have iterated the whole subtree

- `Freq[5]++;`
- `Freq[7]++;`
- `Freq[11]++;`



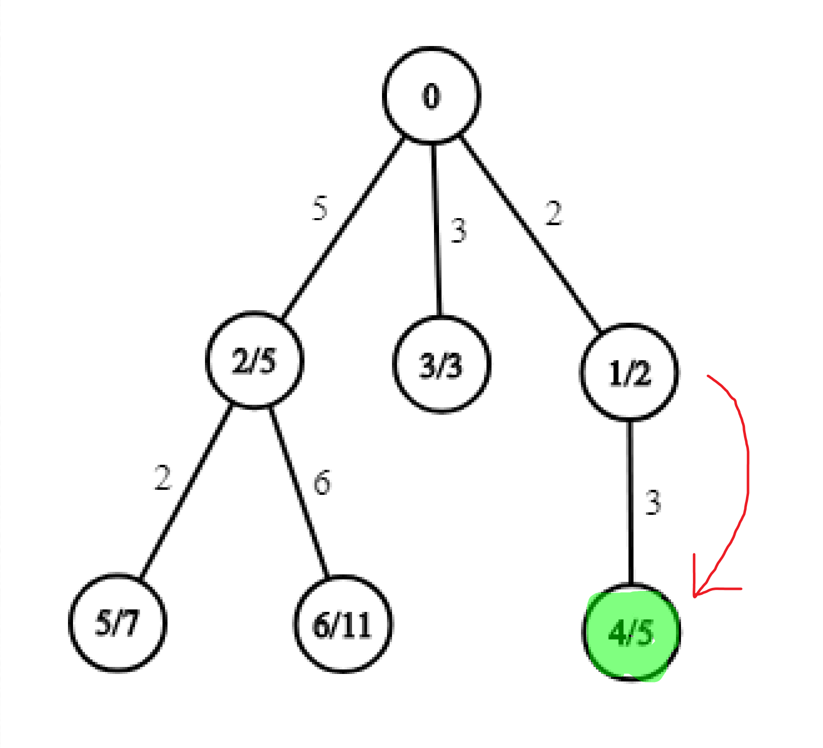
IOI 2011 Race (easier version)

- `Ans += freq[k – 2]`



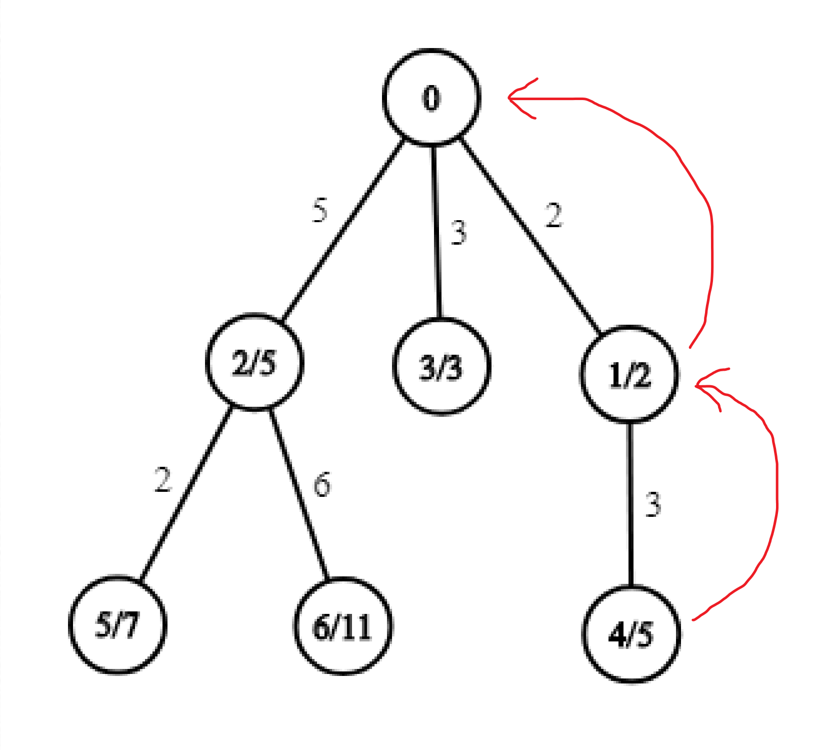
IOI 2011 Race (easier version)

- Ans += freq[k – 5]



IOI 2011 Race (easier version)

- `Freq[2]++;`
- `Freq[5]++;`
- …
- Do the rest yourself
- By this algorithm, we can solve this easier version in O(N)



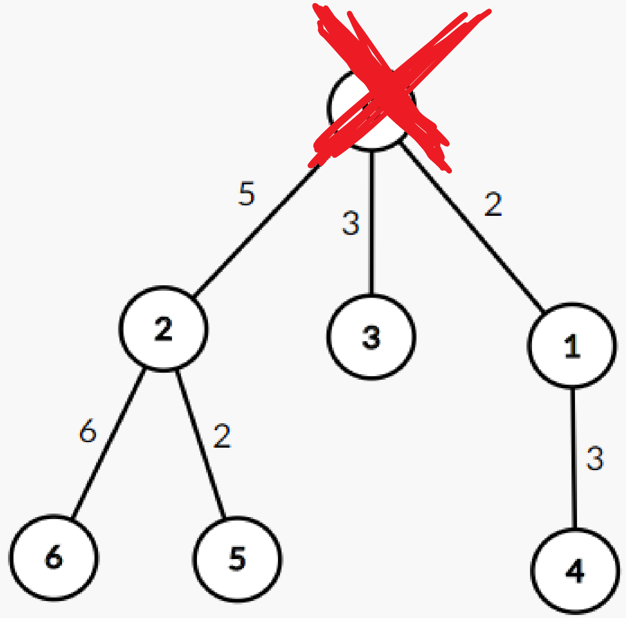
IOI 2011 Race (easier version)

```
void DFS(int x) {
      visit[x] = 1;
      ans += freq[k – dist[x]];
      if (x != 0) update_later.push_back(dist[x]);

      for (auto i: adj node of x) {
            if (visit[i]) continue;
            DFS(i);
            if (x == 0) {  // ----------------------→  This means we have iterated the whole subtree
                  for (auto j : update_later) freq[j]++;
                  update_later.clear();
            }
      }
}
```

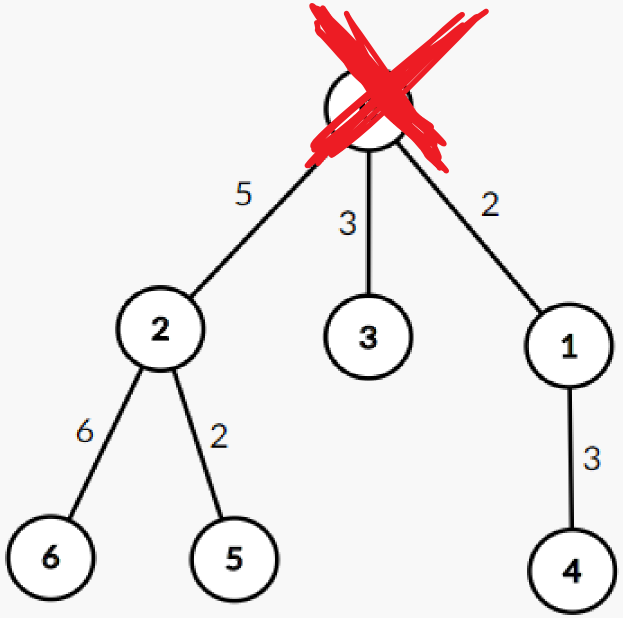
IOI 2011 Race

- Go back to our original problem
- We can iterate all the node, treat it as the root
- Note that when we choose u as the root, run the algorithm before
- Then we have considered ALL the path passing through u
- Which means we can delete node u for later iteration



IOI 2011 Race

- Note that for later iteration, we do not need to iterate all 7 nodes
- E.g. If we treat node as root in this order:
- {0, 2, 3, 1, 6, 5, 4}
- Number of nodes we will access = {7, 3, 1, 2, 1, 1, 1}
- For order {6, 5, 4, 3, 2, 1, 0}
- Number of nodes we will access = {7, 6, 5, 4, 3, 2, 1}



IOI 2011 Race

- In general, if we choose the node in the best order, the total number of node we visit in all DFS trials will be around N lg N
- But in the worst case, the total number of node we visit in all DFS trials will be N * (N - 1) / 2

- What is the best order?

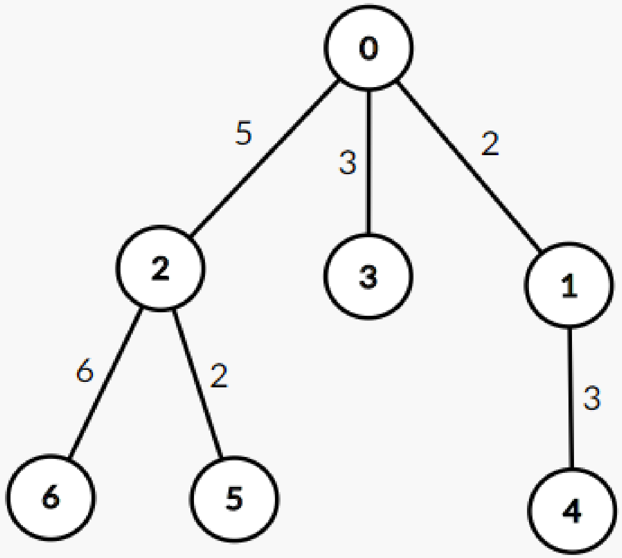
IOI 2011 Race

- The best order is, for each tree in the forest, we should select the Centroid of it as the root each time
- A centroid of a tree with N nodes is a node that after erasing it, all of the remaining component have a size <= N / 2
- Centroid(s) always exist(s) in a tree
- How to find a centroid? → Iterate all node and check the constraint directly

IOI 2011 Race

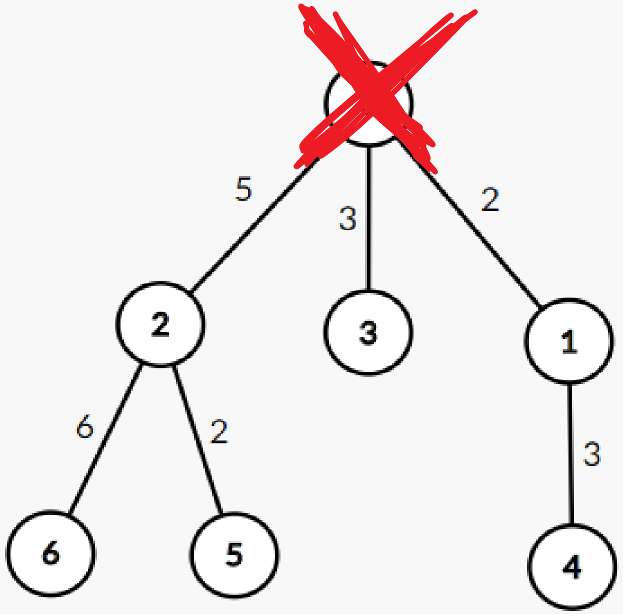
- Centroid = 0
- As the subtree after deleting node 0 is:
- {2, 5, 6}, {3}, {1, 4} → size = {3, 1, 2}

- All subtree size <= 7 / 2 = 3



IOI 2011 Race

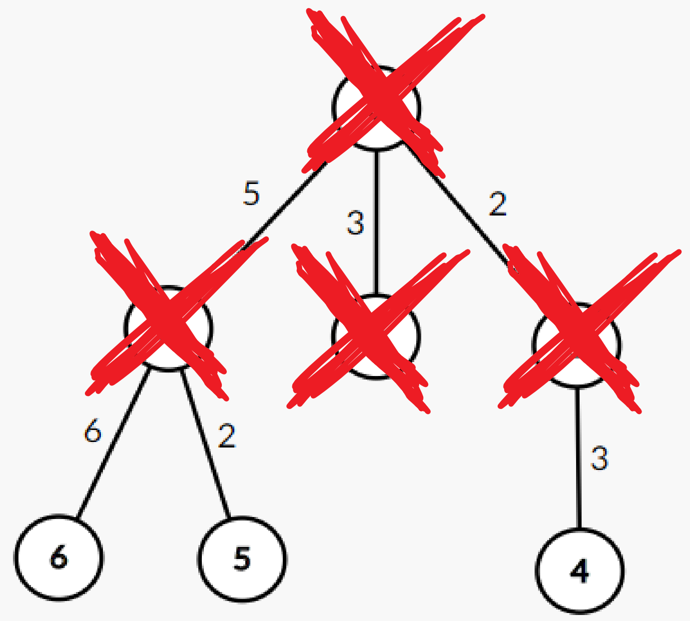
- For the subtree {2, 5, 6} → centroid = 2
- For the subtree {3} → centroid = 3
- For the subtree {1, 4} → centroid = 1 (or 4)



IOI 2011 Race

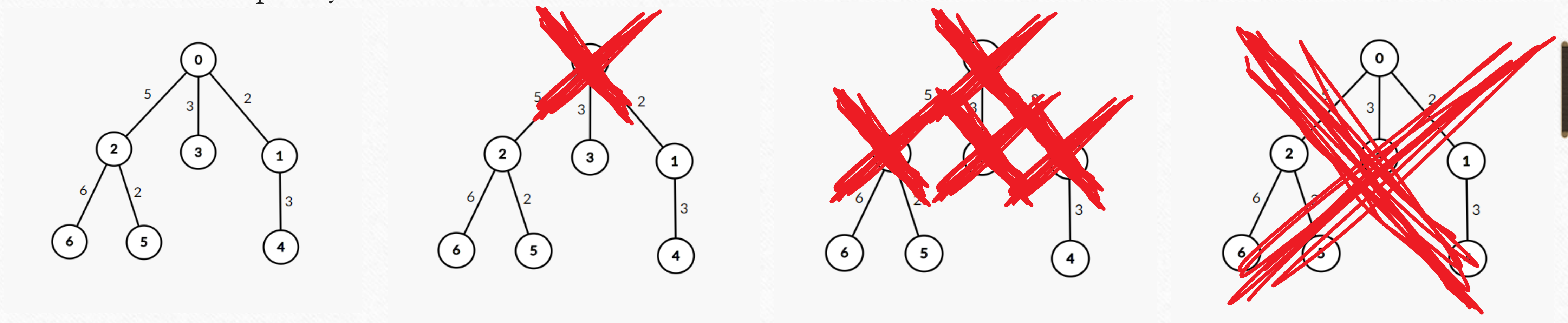
- For the subtree {4} → centroid = 4
- For the subtree {5} → centroid = 5
- For the subtree {6} → centroid = 6

- Done !



IOI 2011 Race

- Time complexity:



IOI 2011 Race

- Time complexity:
- We need 4 layers of deletion to delete all the node
- Note that for each layer, we use O(N) to iterate all the node
- Total number of layer = O(log(N)) as which time we perform a deletion on centroid, the remaining component size decreased by at least half
- Total Time Complexity O(N log N)

CDQ Divide and Conquer

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Contribution Technique

- When we encounter an insert-query problem
- One of the idea is to maintain the inserted element with some data structure, such that we can perform the query efficiently
- Another idea maybe calculate the contribution of each INSERT to each QUERY

Example 1

Alternative Query

CDQ D&C on update/query problem

Alternative Query

- Problem Description:
  - Performing the following operation
  - 1. Insert(x) → Add x in S
  - 2. Query(x) → Count number of value v in S satisfying $v < x$
  - OFFLINE QUERY

Alternative Query

- 7
- Insert(3)
- Insert(5)
- Query(4)
- Query(3)
- Insert(6)
- Insert(2)
- Query(6)

- 1
- 0
- 3

Alternative Query

- You may find this task can be solved by Binary Tree, Segment tree, BIT……
- D&C & contribution technique is another way to solve
- To understand D&C, we may consider an easier version first
  - 1. Insert(x) → Add x in S
  - 2. Query(x) → Count number of value v in S satisfying v < x
  - OFFLINE QUERY
  - All Insert(x) operations are executed before all Query operations

Alternative Query (easy)

- 7
- `Insert(3)`
- `Insert(5)`
- `Insert(6)`
- `Insert(2)`
- `Query(4)`
- `Query(3)`
- `Query(6)`

Alternative Query (Easier version)

- If all Insert(x) go before Query(x)
- We can just simply sort all x in insert(x), binary search / two pointer to answer the query

Alternative Query

- If Insert(x) does not go before all Query(x)
- We can use Divide and Conquer to make Insert(x) go before Query(x)

Alternative Query

- `Insert(3)` → `Op(1)`
- `Insert(5)` → `Op(2)`
- `Query(4)` → …
- `Query(3)`
- `Insert(6)`
- `Insert(2)`
- `Query(6)`

- For each Query() operations, only some Insert() operations need to be considered (those go before that Query)
- Query(4) → Insert(3), Insert(5)
- Query(3) → Insert(3), Insert(5)
- Query(6) → Inst(3), Inst(5), Inst(6), Inst(2)
- To simplify, we may use ID to denote operation
- Op(3) → Op(1), Op(2)

Alternative Query

- `Insert(3)`
- `Insert(5)`
- `Query(4)`
- `---------`
- `Query(3)`
- `Insert(6)`
- `Insert(2)`
- `Query(6)`

- Operation-pairs to consider
- `Op3 → Op{1,2}`
- `Op4 → Op{1,2}`
- `Op7 → Op{1,2,5,6}`

- 1. Divide the operations sequence to half

Alternative Query

- Insert(3)
- Insert(5)
- Query(4)
- ---------
- Query(3)
- Insert(6)
- Insert(2)
- Query(6)

- Operation-pairs to consider
- Op3 → Op{1,2}
- Op4 → Op{1,2}
- Op7 → Op{1,2,5,6}

- 1. Divide the operations sequence to half
- 2. Consider Insert() in first part and Query() in second part only

Alternative Query

- Insert(3)
- Insert(5)
- ---------
- Query(3)
- Query(6)

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- Op3 → Op{1,2}
- Op4 → Op{1,2}
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- 1. Divide the operations sequence to half
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Note that now, the operations sequence become a Insert-first-sequence

Alternative Query

- `Insert(3)`
- `Insert(5)`
- `---------`
- `Query(3)`
- `Query(6)`

- Operation-pairs to consider
- `Op3 → Op{1,2}`
- `Op4 → Op{1,2}`
- `Op7 → Op{1,2,5,6}`

- `Ans(Op3) → 0`
- `Ans(Op4) → 0`
- `Ans(Op7) → 2`

- 1. Divide the operations sequence to half
- 2. Consider Insert() in first part and Query() in second part only

Note that now, the operations sequence become a Insert-first-sequence

- 3. Use the solution of easy version to solve this scenario

Alternative Query

- Insert(3)
- Insert(5)
- ---------
- Query(3)
- Query(6)

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- Op3 → Op{1,2}
- Op4 → Op{1,2}
- Op7 → Op{1,2,5,6}

- Ans(Op3) → 0
- Ans(Op4) → 0
- Ans(Op7) → 2

- 1. Divide the operations sequence to half
- 2. Consider Insert() in first part and Query() in second part only

Note that now, the operations sequence become a Insert-first-sequence

- 3. Use the solution of easy version to solve this scenario
- 4. Note that the operation-pair highlighted in blue is what we have calculated

Alternative Query

- Insert(3)
- Insert(5)
- Query(4)
- ---------
- Query(3)
- Insert(6)
- Insert(2)
- Query(6)

- Operation-pairs to consider
- Op3 → Op{1,2}
- Op4 → Op{1,2}
- Op7 → Op{1,2,5,6}

- Ans(Op3) → 0
- Ans(Op4) → 0
- Ans(Op7) → 2

- Note that the operation-pair highlighted in blue is what we have calculated
- What we have NOT calculated is the operation-pairs that Both operations in the pair belongs to a single part only
- So, what we should do is to apply the above algorithm to the first half, second half respectively

Alternative Query

- 1st part only
- Insert(3)
- Insert(5)
- ---------
- Query(4)

- Operation-pairs to consider
- Op3 → Op{1,2}
- Op4 → Op{1,2}
- Op7 → Op{1,2,5,6}
- Ans(Op3) → 1
- Ans(Op4) → 0
- Ans(Op7) → 2

- Recursively do the first part
- Again, divide into 2 half and consider Insert in first, query in second only
- The red part are the pairs we calculated in this scenario
- Again, then do it recursively…

Alternative Query

- `Insert(3)`
- `---------`
- `Insert(5)`

- Operation-pairs to consider
- `Op3 → Op{1,2}`
- `Op4 → Op{1,2}`
- `Op7 → Op{1,2,5,6}`

- `Ans(Op3) → 1`
- `Ans(Op4) → 0`
- `Ans(Op7) → 2`

- No update in this scenario

Alternative Query

- `Query(4)`

- Operation-pairs to consider
- Op3 → Op{1,2}
- Op4 → Op{1,2}
- Op7 → Op{1,2,5,6}

- Ans(Op3) → 1
- Ans(Op4) → 0
- Ans(Op7) → 2

- No update in this scenario

Alternative Query

- 2^{nd} part
- Query(3)
- Insert(6)
- ---------
- Insert(2)
- Query(6)

- Operation-pairs to consider
- Op3 → Op{1,2}
- Op4 → Op{1,2}
- Op7 → Op{1,2,5,6}
- Ans(Op3) → 1
- Ans(Op4) → 0
- Ans(Op7) → 2

- Again, divide into 2 half and consider Insert in first, query in second only
- The red part are the pairs we calculated in this scenario
- Again, then do it recursively…

Alternative Query

- `Query(3)`
- `---------`
- `Insert(6)`

- Operation-pairs to consider
- `Op3 → Op{1,2}`
- `Op4 → Op{1,2}`
- `Op7 → Op{1,2,5,6}`

- `Ans(Op3) → 1`
- `Ans(Op4) → 0`
- `Ans(Op7) → 2`

- No update in this scenario

Alternative Query

- Insert(2)
- ---------
- Query(6)

- Operation-pairs to consider
- Op3 → Op{1,2}
- Op4 → Op{1,2}
- Op7 → Op{1,2,5,6}

- Ans(Op3) → 1
- Ans(Op4) → 0
- Ans(Op7) → 3

- Done !!!

Framework

- Let solve(1, n) be the procedure to solve the Online query problem

```
Void solve(int l, int r) {
    Int mid = (l + r) / 2;
    Insert_list = Extract_Insert(l, mid);
    Query_list = Extract_Query(mid + 1, r);
    Solve_Insert_First_Query_easier_version(Insert_list, Query_list);
    If (mid – l > 1) Solve(l, mid);
    If (r – (mid + 1) > 1) Solve(mid + 1, r);
}
```

Alternative Query

- Time Complexity:
- Let T(n) = Time complexity of
  `Solve_Insert_First_Query_easier_version(Insert_list, Query_list);`

  `Where n = sum of size of the two list`
- Note that we will call
- `solve(1, 8)` → `solve(1, 4) + solve(4, 8)` → `solve(1, 2) + solve(3, 4)` ….
- Like a merge sort, this recursive calling give a lg(n) factor
- i.e. Time complexity = T(n) lg(n)
- For the algorithm above, T(n) = O(n lg n) → Time complexity = n lg(n) lg(n)

Example 2

Inversion

Inversion

- Problem Description:
  - Given an array A[1..n], find the number of inversion of the sequence
  - Inversion: a pair (x, y) (1 <= x, y <= n) where x < y and A[x] > A[y]
- E.g. [1, 3, 2, 5, 4]
  - Inversion: (3, 2), (5, 4) → 2

Inversion

- Solution:
- You can treat it to a insert-query problem
- Iterate the array from the beginning to the end
  - query the number of elements in S greater than A[i]
  - Inserting A[i] to S

Inversion

- E.g. A[] = [3, 2, 5, 4]
- Query(3)
  Insert(3)
  Query(2)
  Insert(2)
  Query(5)
  Insert(5)
  Query(4)
  Insert(4)
- Sum of all query answer is the number of inversion

Inversion

- So, we can transform it to insert-query and use CDQ D&C to solve it

Example 3

Longest Increasing Subsequence

Longest Increasing Subsequence

- Sometimes, DP problem can also be solved by D&C → (Usually, CHT dp)
- Consider LIS as an example:
- Problem Description:
  - Given an array A[1..n]
  - Find the LIS of it
- Example:
- A = [2, 5, 3, 4, 7, 5, 6]
- LIS is[2, 3, 4, 5, 6] → 5

Longest Increasing Subsequence

- Naïve solution is an O(N^2) dp:
-
```
for (int i = 0; i < n; i++) {
    dp[i] = 1;
    for (int j = 0; j < i; j++)
      if (A[j] < A[i] && dp[j] + 1 > dp[i])
          dp[i] = dp[j] + 1;
}
```

Longest Increasing Subsequence

- In fact, we can model it to an insert-query problem
- Insert a pair(A[i], dp[i]) into a set S
- Query(j) is to find a pair in S where (A[i] < A[j]) and dp[i] is maximum
- [2, 5, 3, 4, 7, 5, 6] can be remodel as
- Query(2) → Insert(2) → Query(5) → Insert(5) ……

Longest Increasing Subsequence

- Let's think back to CDQ D&C
- What if all insert are before query, can we solve it much easier?
- Insert(A[1], dp[1]) → Insert(A[2], dp[2]) → Insert(A[3], dp[3]) → Query(A[4]) → Query(A[5])

- Yes!! We can sort the insert & query according to A[i]
- For each insert, we perform bestans = max(bestans, dp[i] + 1);
- For each query, we set dp[j] as bestans

Longest Increasing Subsequence

- Insert(A[1], dp[1]) → Insert(A[2], dp[2]) → Insert(A[3], dp[3]) → Query(A[4]) → Query(A[5])
- Let A = {4, 5, 1, 2, 6}
- dp[1] = 1, dp[2] = 2, dp[3] = 1
- Sort according to A first
- Insert(A[3], dp[3]) → Query(A[4]) → Insert(A[1], dp[1]) → Insert(A[2], dp[2]) → Query(A[5])
- Query(A[4]) → bestans = 2 at that time
- Query(A[5]) → bestans = 3 at that time

Longest Increasing Subsequence

- Insert(A[1], dp[1]) → Insert(A[2], dp[2]) → Insert(A[3], dp[3]) → Query(A[4]) → Query(A[5])
- Note that we just consider the contribution of insert(1 to 3) to query(4 to 5)
- We should also consider the contribution of insert(4) to query(5) as well
- Same as the example above, CDQ D&C help us to solve it!

Longest Increasing Subsequence

- Insert(A[1], dp[1]) → Insert(A[2], dp[2]) → Insert(A[3], dp[3]) → Query(A[4]) → Query(A[5])
- Wait, however how does we know the value of dp[1], dp[2], dp[3] when we solving the above instance

Longest Increasing Subsequence

```
Void solve(int l, int r) {
    Int mid = (l + r) / 2;
    If (mid – l > 1) Solve(l, mid);  → solve the 1st half instance first to get the value of dp[1-3] first
    Insert_list = Extract_Insert(l, mid);
    Query_list = Extract_Query(mid + 1, r);
    Solve_Insert_First_Query_easier_version(Insert_list, Query_list);
    If (r – (mid + 1) > 1) Solve(mid + 1, r);
}
```

CDQ D&C

- When we encounter a insert-query type problem and you found that it is easier for us to solve the insert-first-query-last version → use CDQ D&C
- Sometimes the problem may not explicitly tell you what is the insert and what is the query, but we may able to remodel it to insert & query style

What kind of insert-query can be solved by D&C?

- 1. The query must to be OFFLINE
- 2. The Insert operation should be independent
  - E.g. If we have delete operation, we may not able to solve it by CDQ D&C
- 3. The Query operation should be able to solved by contribution technique
  - E.g. If we are going to query the median, we may not able to solve it by CDQ D&C
  - Because median cannot be found by considering the contribution of each insert one-by-one

Summary

Finally

- D&C often help you solve Data Structure problem (e.g. insert-query / range query problem)
- With the help of D&C, we usually able to figure out a algorithm to get rid of using advanced data structure (2D segment → segment tree) or (Segment tree → array / 2 pointer)
- D&C usually run in good constant time!

Practice Problem

- CDQ D&C:
- UVaLive 5871
- UVaLive 6374
- CEOI 2017 day-2 Building Bridges (can be found in CSAcademy)

- Centroid Decomposition
- IOI 2011 Race
- UVaLive 7148
- CSAcademy Round 58 – Path-Investions

Appendix 1

CDQ + Line sweeping + Segment tree example
Arnook's Defensive Line

UVaLive 5871 - Arnook's Defensive Line

- Problem Description:
  - Performing the following operation
  - 1. Insert(l, r) → Add a segment [l, r] in S
  - 2. Query(l, r) → Count number of segment [a, b] in S satisfying `a <= l && r <= b`
  - OFFLINE QUERY

UVaLive 5871 - Arnook's Defensive Line

- Solution 1:
  - 2D segment tree → O(n lg n lg n)
  - Drawbacks: Hard to implement, Large constants
- Solution 2:
  - The question is similar to the last question, except what is inserting & what is querying
  - If we can solve the "insert-first-query-then" version, we can use CDQ D&C to solve it

UVaLive 5871 - Arnook's Defensive Line

- Consider a simpler problem
  - All query command appear after all insert command
- Solution:
  - Sweeping line + 1d segment tree (dynamic / discretize)
  - Sort the query and segment according to the right bound
  - Insert the left bound of the segment when we sweep to the right bound of it
  - Query sum in [1, x] when we sweep to the right bound of a query

Framework

- You can use the same framework, just change the Solve_Insert_First…… part

```
Void solve(int l, int r) {
    Int mid = (l + r) / 2;
    Insert_list = Extract_Insert(l, mid);
    Query_list = Extract_Query(mid + 1, r);
    Solve_Insert_First_Query_easier_version(Insert_list, Query_list);
    If (mid - l > 1) Solve(l, mid);
    If (r - (mid + 1) > 1) Solve(mid + 1, r);
}
```

UVaLive 5871 - Arnook's Defensive Line

- Time Complexity:
- Let T(n) = Time complexity of
  `Solve_Insert_First_Query_easier_version(Insert_list, Query_list);`
- For the algorithm above, T(n) = O(n lg n) → Time complexity = n lg(n) lg(n)
- Same as 2d segment tree but smaller constant and way easier to implement