HKOI 2018/19 Solution
S192 - Two Towers

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Background

- Given an $N \times M$ grid.
- Want to build two communications towers (A and B).
- Tower A's power $= P_A$; Tower B's power $= P_B$.
- Signal strength of tower A at cell X is

$$Str_A(X) := \max(0, P_A - MD(\text{tower A}, X)).$$

- MD: Manhattan distance.
  $MD((r_1, c_1), (r_2, c_2)) = |r_2 - r_1| + |c_2 - c_1|$.

4	3	2	3	4	5	6
3	2	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	2	3	4	5

MD between green cell and other cells

Objective

- Build towers at appropriate positions to maximize the signal strength at the worst cell (household).
- Formally, want to maximize

$$\min_{X} \left[\max(Str_A(X), Str_B(X))\right]$$

Samples

4(B)	5	6	5	4
5	6	7(A)	6	5
6	5	6	5	4

Sample 1: P_A = 7, P_B = 0

8	9	8
9	10(B)	9
8	9	8
9	10(A)	9
8	9	8

Sample 2: P_A = 10, P_B = 10

1	1(A)
2	1
3(B)	2
2	1

Sample 5: P_A = 1, P_B = 3

Subtasks and Stats

SUBTASKS

For all cases:
$1 \leq N, M \leq 5 \times 10^8$
$0 \leq P_A, P_B \leq 10^9$

	Points	Constraints
1	10	$P_A = 0$
2	14	$N = 1$
3	15	$N = 2$, $M \geq 2$
4	16	$1 \leq N, M \leq 10$
5	22	$1 \leq N, M \leq 2000$
6	23	No additional constraints

Task	Attempts	Max	Mean	Std Dev	Subtasks					
S192 - Two Towers	67	40	10.97	10.889	10: 44	14: 4	15: 1	16: 14	22: 0	23: 0

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Subtask 1 (10 pts): Tower A is useless

- Intuitively, the corner cells are hardest to reach.
- Place tower B in the middle, e.g. cell $(\lceil \frac{N}{2} \rceil, \lceil \frac{M}{2} \rceil)$.
- Distance to farthest cell $= \lfloor \frac{N}{2} \rfloor + \lfloor \frac{M}{2} \rfloor$.
  Examples:

5			2				6
			1				
3	2	1	B	1	2	3	4
			1				
5			2				6

N = 5, M = 8

3		1			4
2	1	B	1	2	3
		1			
4		2			5

N = 4, M = 6

- So the answer is $\max(0, P_B - \lfloor \frac{N}{2} \rfloor - \lfloor \frac{M}{2} \rfloor)$.

Subtask 4 (16 pts): N and M are as small as 20

- Well, just **exhaust** the $(NM)^2$ possible positions of towers A and B :)
- Checking can be done in $O(NM)$.
- Write six nested loops, get 16 points. What a bargain!
- Time complexity: $O(N^3M^3)$.

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Assumptions

Assume:

- $N \leq M$,
- $P_A \leq P_B$.

Idea 1: Optimization $\longrightarrow$ Feasibility

- Let

$$Good(V) = \begin{cases} 1, & \text{if an answer of V is attainable;} \\ 0, & \text{otherwise.} \end{cases}$$

- Then:

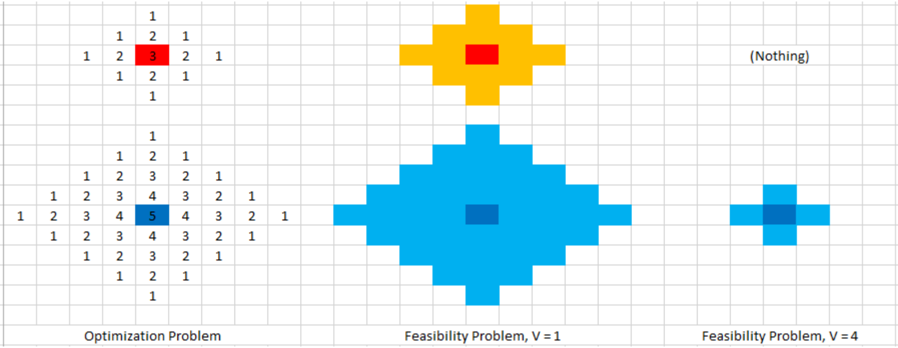
$$Good(0) = \cdots = Good(Answer) = 1;$$
$$Good(Answer + 1) = Good(Answer + 2) = \ldots = 0.$$

- Problem transformation (for $V > 0$):

Feasibility Problem (Finding $Good(V)$)

Given $rad_A := P_A - V$, $rad_B := P_B - V$ (can be -ve).
Determine positions for towers A and B so that, for all cells X, either $MD(\text{tower A}, X) \leq rad_A$ or $MD(\text{tower B}, X) \leq rad_B$.

Illustration



Becomes a **covering by tilted squares** problem.

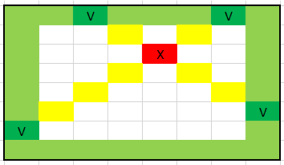
Idea 2: Enough to cover all boundary cells

We have this powerful and surprising observation:

Theorem 1

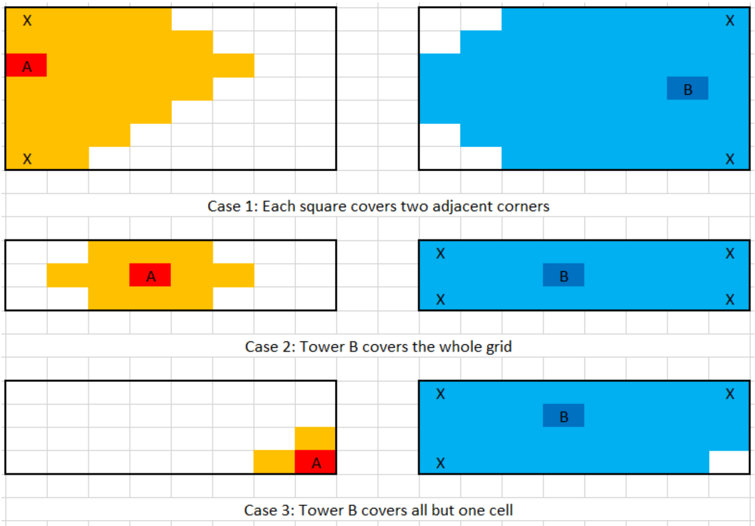
For two (in fact ≤ 3) tilted squares covering a rectangular grid, covering all boundary cells implies covering the whole grid!!!

Here is a simple proof (proof by contradiction).



Each tilted square can only cover **one** V cell, if cell X cannot be covered.

Idea 3: Three configurations



Idea 3: Three configurations

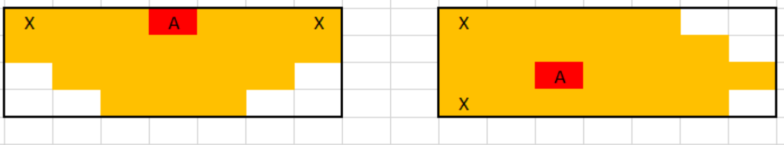
- Need to check whether a covering is possible, using one of the configurations.
- Cases 2 and 3 are easy! Now we focus on case 1.

Idea 4: Greedy, greedy, greedy

Greedy Idea 1

Prefer covering short (horizontal) edge to covering long (vertical) edge.

Illustration:



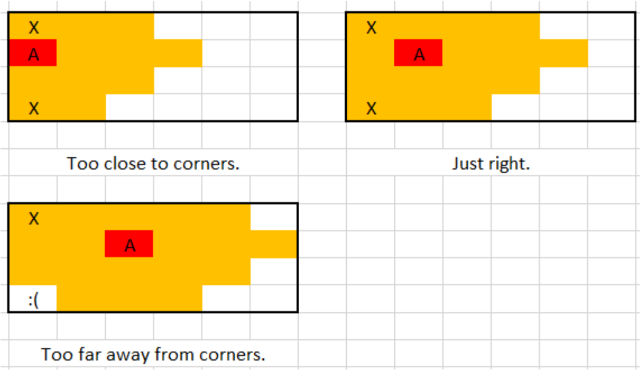
In the right configuration, it takes a **smaller** tilted square to cover the rest.

Idea 4: Greedy, greedy, greedy

Greedy Idea 2

Go as far from corners **as possible**.

Illustration:

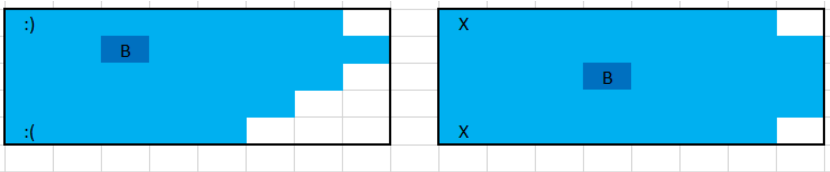


Idea 4: Greedy, greedy, greedy

Greedy Idea 3

Prefer closer to the middle row.

Illustration:



In the left configuration, the :(corner is **"pulling" B to the left**, limiting its reach.

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Piecing them all together

Recall our beautiful ideas:

1. Optimization $\longrightarrow$ Feasibility
2. Enough to cover all boundary cells
3. Three configurations
4. Greedy, greedy, greedy

Below we describe an $O(\log RANGE)$ solution.

Remember the assumptions $N \leq M$ and $P_A \leq P_B$.

Step 1: Binary search on answer

CAUTION: There are so many binary search styles. Convert below to your favourite one.

Set $L := 0$, $R := 10^9 + 1$
While $L + 1 < R$
  Set $V := \lfloor \frac{L+R}{2} \rfloor$
  If $Good(V)$ then set $L := V$
  Else set $R := V$

Set $Ans := L$
Run $Good(Ans)$ to get tower positions
Output answer and tower positions

Step 2: Handle easy cases

If $V = 0$
  Set $A = B = (1, 1)$
  Return `True`

Set $rad_A := P_A - V$, $rad_B := P_B - V$

If $rad_B \geq \lfloor \frac{N}{2} \rfloor + \lfloor \frac{M}{2} \rfloor$
  Case 2 works!
  Set $A = (1, 1), B = (\lceil \frac{N}{2} \rceil, \lceil \frac{M}{2} \rceil)$
  Return `True`

If (N and M are even) AND ($rad_B \geq \frac{N}{2} + \frac{M}{2} - 1$) AND ($rad_A \geq 0$)
  Case 3 works!
  Set $A = (N, M), B = (\frac{N}{2}, \frac{M}{2})$
  Return `True`

Step 3: Check if short edges can be covered

If $rad_A < \lfloor \frac{N}{2} \rfloor$
    Return `False`

Step 4: Choose best places for towers

Suppose $A = (r_A, c_A)$, $B = (r_B, c_B)$.

Set $r_A := \lceil \frac{N}{2} \rceil$, $r_B := \lceil \frac{N+1}{2} \rceil$ (Greedy Ideas 1 + 3)
Set $c_A := \min(M, 1 + (rad_A - \lfloor \frac{N}{2} \rfloor))$
Set $c_B := \max(1, M - (rad_B - \lfloor \frac{N}{2} \rfloor))$ (Greedy Idea 2)

Step 5: Check if top and bottom rows are covered

For row 1:

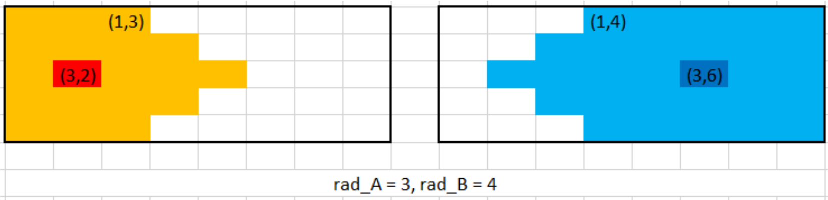
- Rightmost cell covered by A is $c_A + (rad_A - (r_A - 1))$
- Leftmost cell covered by B is $c_B - (rad_B - (r_B - 1))$

For row N:

- Rightmost cell covered by A is $c_A + (rad_A - (N - r_A))$
- Leftmost cell covered by B is $c_B - (rad_B - (N - r_B))$

All cells covered $\iff$ $rightA \geq leftB - 1$.
By symmetry, it is enough to check row 1.

Example 1 (for Steps 4 - 5)



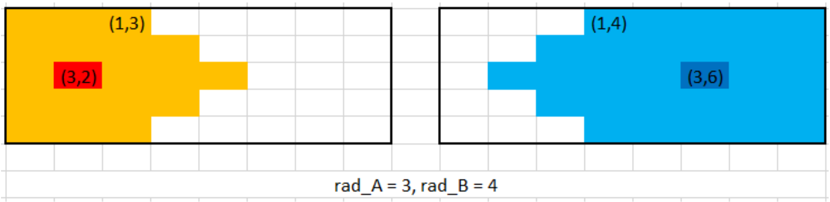
Set $r_A := \lceil \frac{N}{2} \rceil = \lceil \frac{5}{2} \rceil = 3$

Set $r_B := \lceil \frac{N+1}{2} \rceil = \lceil \frac{5+1}{2} \rceil = 3$

$\lfloor \frac{N}{2} \rfloor = \lfloor \frac{5}{2} \rfloor = 2$

$$\begin{aligned} \text{Set } c_A &:= \min(M, 1 + (rad_A - \lfloor \tfrac{N}{2} \rfloor)) \\ &= \min(8, 1 + (3 - 2)) &= 2 \\ \text{Set } c_B &:= \max(1, M - (rad_B - \lfloor \tfrac{N}{2} \rfloor)) \\ &= \max(1, 8 - (4 - 2)) &= 6 \end{aligned}$$

Example 1 (cont'd)

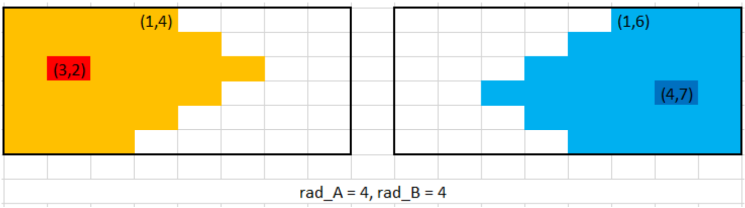


For row 1:

- $rightA := c_A + (rad_A - (r_A - 1)) = 2 + (3 - (3 - 1)) = 3$
- $leftB := c_B - (rad_B - (r_B - 1)) = 6 - (4 - (3 - 1)) = 4$

Indeed $rightA \geq leftB - 1$. Return `True`.

Example 2 (for Steps 4 - 5)



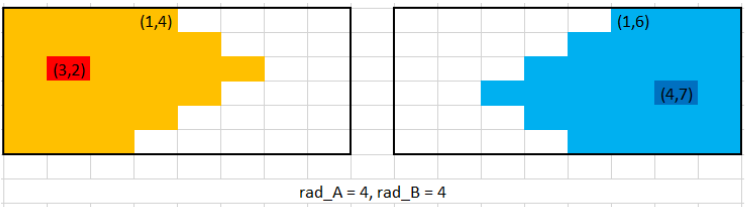
Set $r_A := \lceil \frac{N}{2} \rceil = \lceil \frac{6}{2} \rceil = 3$

Set $r_B := \lceil \frac{N+1}{2} \rceil = \lceil \frac{6+1}{2} \rceil = 4$

$\lfloor \frac{N}{2} \rfloor = \lfloor \frac{6}{2} \rfloor = 3$

$$\begin{aligned} \text{Set } c_A &:= \min(M, 1 + (rad_A - \lfloor \tfrac{N}{2} \rfloor)) \\ &= \min(8, 1 + (4 - 3)) && = 2 \\ \text{Set } c_B &:= \max(1, M - (rad_B - \lfloor \tfrac{N}{2} \rfloor)) \\ &= \max(1, 8 - (4 - 3)) && = 7 \end{aligned}$$

Example 2 (cont'd)



For row 1:

- $rightA := c_A + (rad_A - (r_A - 1)) = 2 + (4 - (3 - 1)) = 4$
- $leftB := c_B - (rad_B - (r_B - 1)) = 7 - (4 - (4 - 1)) = 6$

Return `False` since $rightA < leftB - 1$.

Remark

1. An $O(1)$ solution exists. Instead of binary search, just directly solve inequalities arising from the covering conditions. (Nasty!)
2. To make the arguments in **Idea 4: Greedy, greedy, greedy** rigorous, again you need to write down inequalities. Proof is left as exercise. (Nasty!!)