J191 Alice and Wings

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Basic Notations – Modular Arithmetic

$$remainder\ of\ x\ divided\ by\ M = x\ mod\ M$$

(C/C++: $= x\ \%\ M$)

$$x\ mod\ M = y\ mod\ M$$
$$\Rightarrow x \equiv y\ (mod\ M)$$

For simplicity,

all modular equations
in this PowerPoint
are
in $mod\ M$

Basic Notations – Modular Arithmetic

$$x \equiv y$$

$\Rightarrow$ There exists some integer k such that

$$x = y + kM$$

Equivalently,

$$\Rightarrow x - y = kM$$

$$\Rightarrow x - y \equiv 0$$

Basic Notations – Modular Arithmetic

$$x \equiv y$$
$$\Rightarrow x \pm z \equiv y \pm z$$

$$a \equiv c$$
$$b \equiv d$$
$$\Rightarrow a \pm b \equiv c \pm d$$

Basic Notations – Modular Arithmetic

$$x \equiv y$$
$$0 \leq x, y < M$$

$$\Rightarrow x = y$$

Simple proof:

$$0 \leq x, y < M$$
$$-M = 0 - M < x - y < M - 0 = M$$
$$x - y = kM \Rightarrow -1 < k < 1, k = 0$$
$$x - y = 0, x = y$$

Problem Statement

- On day i, send i boxes of A Wings + 1 box of B wings
  - $(A\ i + B)$ Wings in total
  - $A = daily\ increase$
  - $B = fixed\ amount$
- Alice groups them into packs of M wings, and brings the remaining fewer-than-M Wings home.
  - $W_i = (A\ i + B)\ mod\ M$ Wings were brought back home

Problem Statement

- Given W_1, W_2, W_3, determine A, B, M
- $0 \leq W_1, W_2, W_3, A, B < M$
- Multiple solutions => minimize M: 100% marks, otherwise 50% marks

Example 1 – Sample 3

- $A = 2, B = 3, M = 6$
- $W_1 = (2 \times 1 + 3)\ mod\ 6 = 5$
- $W_2 = (2 \times 2 + 3)\ mod\ 6 = 1$
- $W_3 = (2 \times 3 + 3)\ mod\ 6 = 3$

SUBTASKS

For all cases: $0 \leq W_1, W_2, W_3 \leq 3 \times 10^8$

	Points	Constraints
1	4	$W_1 = W_2$
2	12	$0 \leq W_1 \leq W_2 \leq W_3$ or $W_1 \geq W_2 \geq W_3 \geq 0$
3	20	$0 \leq W_1, W_2, W_3 \leq 100$
4	28	$0 \leq W_1, W_2, W_3 \leq 3000$
5	36	No additional constraints

SAMPLE TESTS

	Input	Output
1	2 4 6	2 0 7

This sample scores 100% of the points

2	2 4 6	2 0 2018

This sample only scores 50% of the points

3	5 1 3	2 3 6

This corresponds to the example in the problem statement.

4	2 6 9	No solution
5	3 5 0	2 1 7
6	6 0 13	13 12 19
7	27 57 87	30 85 88

Attempts	Max	Mean	Std Dev	Subtasks				
75	100	11.493	23.598	4: 36	12: 14 6: 2	20: 12	28: 5 14: 1	36: 4

Attempts	Max	Mean	Std Dev	Subtasks				
75	100	11.493	23.598	4: 36	12: 14 6: 2	20: 12	28: 5 14: 1	36: 4

- Out of 75 contestants who have attempted this task,

  38 people got 0 marks

  $38 > 37.5$

- Out of 17 Silver award recipients,

  8 people got 0 marks + 1 person did not attempt

  $9 > 8.5$

Subtask 1: $W_1 = W_2$

$$W_1 \equiv A + B$$
$$W_2 \equiv 2A + B$$

$$W_2 - W_1 \equiv (2A + B) - (A + B)$$
$$0 \equiv A$$

Together with

$$0 \leq A < M$$

we have

$$A = 0$$

Subtask 1: $W_1 = W_2$

$$W_i \equiv Ai + B \equiv 0(i) + B \equiv B$$

Together with

$$0 \leq W_i, B < M$$

we have

$B = W_1 = W_2$ (guaranteed)

$B = W_3$ as well (to be checked)

Subtask 1: $W_1 = W_2$

- During competition, you may not do all those calculations.
- Intuitively, since $W_1 = W_2$,

$$\text{Daily increase } A = 0$$

$$\text{Fixed amount } B = W_1 (= W_2)$$

- Also intuitively, W_3 should be equal to W_1 and W_2 in the previous two days.

Subtask 1: $W_1 = W_2$

- So the solution is simple:
- **CHECK** if $W_3 = W_2$
  - No $\Rightarrow No\ solution$
  - Yes $\Rightarrow A = 0, B = W_1$
- One last question: What is the minimal M?

Subtask 1: $W_1 = W_2$

- Recall that $0 \leq W_i, B < M$
- Simply set $M = W_1 + 1$

Solution 1 – Subtask 1 only

PSUEDOCODE

```
INPUT W1, W2, W3
IF(W1==W2 && W2==W3)
 A = 0, B = W1,
 M = W1 + 1
 PRINT A, B, M
ELSE PRINT ”No
solution”
```

Subtask	Score	Max Score
1	4	4
2	0	12
3	0	20
4	0	28
5	0	36
Total	**4**	**100**

Subtask 2: $W_1 \leq W_2 \leq W_3$ / $W_1 \geq W_2 \geq W_3$

Define

$$d_1 = W_2 - W_1$$
$$d_2 = W_3 - W_2$$

, d for difference

$$d_i \equiv W_{i+1} - W_i \equiv (A(i+1) + B) - (Ai + B) \equiv A$$

KEY: $d_1 \equiv d_2 \equiv A$

$$0 \leq W_i < M$$
$$\Rightarrow -M < d_1, d_2 < M$$

Subtask 2: $W_1 \leq W_2 \leq W_3$ / $W_1 \geq W_2 \geq W_3$

Since

$$W_1 \leq W_2 \leq W_3 \; or \; W_1 \geq W_2 \geq W_3$$

, d_1 and d_2 are either both positive or both negative.

Together with

$$-M < d_1, d_2 < M$$

We have

$$-M < d_1, d_2 \leq 0 \; or \; 0 \leq d_1, d_2 < M$$

$$d_1 = d_2$$

Subtask 2: $W_1 \leq W_2 \leq W_3$ / $W_1 \geq W_2 \geq W_3$

Remember that

$$d_i \equiv A$$

as well as

$$-M < d_i < M, 0 \leq A < M$$

so A and B can be determined by

$$A = (d_1 + M) \,\%\, M$$
$$B = (W_1 - A + M) \,\%\, M$$

Subtask 2: $W_1 \leq W_2 \leq W_3$ / $W_1 \geq W_2 \geq W_3$

- So the solution is simple:
- **CHECK** if $d_1 = d_2$
  - No ⇒ $No\ solution$
  - Yes ⇒ $A = (d_1 + M)\ \%\ M, B = (W_1 - A + M)\ \%\ M$
- One last question: What is the minimal M?

Subtask 2: $W_1 \leq W_2 \leq W_3$ / $W_1 \geq W_2 \geq W_3$

- Recall that $0 \leq W_i, B < M$
- Simply set $M = \max(W_i) + 1$

Solution 2 – Subtasks 1 and 2 only

PSUEDOCODE

```
INPUT W1, W2, W3
d1 = W2 - W1
d2 = W3 - W2
IF(d1 == d2)
 M = max(Wi) + 1
 A = (d1 + M) % M
 B = (W1 - A + M) % M
 PRINT A, B, M
ELSE PRINT "No solution"
```

Subtask	Score	Max Score
1	4	4
2	12	12
3	0	20
4	0	28
5	0	36
Total	**16**	**100**

	$W_1 \leq W_2$	$W_1 \geq W_2$
$W_2 \leq W_3$	Done Subtask 2	**Pending**
$W_2 \geq W_3$	**Pending**	Done Subtask 2

Following this direction,
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Why not EXHAUSTION

Subtask 3: $0 \leq W_1, W_2, W_3 \leq 100$

- Exhaust M, A, and B
- 3 for-loops
- Check if $W_i = (Ai + B) \% M$

~ THE END ~

Subtask 4: $0 \leq W_1, W_2, W_3 \leq 3000$

- No need to exhaust B
- B can be determined by

$$B = (W_1 - A + M) \,\%\, M$$

- 2 for-loops only

~ THE END ~

Solution 3 – Subtask 3 only

PSUEDOCODE

```
INPUT W1, W2, W3
FOR M: 1 - 100
 FOR A: 0 - M-1
  FOR B: 0 - M-1
   IF(W1==(A+B)%M &&
W2==(2A+B)%M &&
W3==(3A+B)%M)
    PRINT A, B, M
    END
PRINT ”No solution”
```

Solution 4 – Subtasks 3 and 4 only

PSUEDOCODE

```
INPUT W1, W2, W3
FOR M: 1 - 3000
 FOR A: 0 - M-1
  B = (W1 - A + M) % M
  IF(W2==(2A+B)%M &&
W3==(3A+B)%M)
   PRINT A, B, M
   END
PRINT ”No solution”
```

Solution 3 – Subtask 3 only

Subtask	Score	Max Score
1	0	4
2	0	12
3	0	20
4	0	28
5	0	36
Total	**0**	**100**

Solution 4 – Subtasks 3 and 4 only

Subtask	Score	Max Score
1	0	4
2	0	12
3	0	20
4	0	28
5	0	36
Total	**0**	**100**

WHY

Example 2

- $W_1 = 1, W_2 = 0, W_3 = 1$
- Looks easy: odd, even, odd
- $A = 1, B = 0, M = 2$

Example 2*

- Multiple Example 2 by 3000

- $W_1 = 3000, W_2 = 0, W_3 = 3000$
- $\Rightarrow A = 3000, B = 0, M = 6000$

IMPORTANT NOTE:

$$0 \leq W_i \leq n$$

DOES **NOT** IMPLY

$$0 \leq M \leq n$$

Subtask 3: $n = 100$, Subtask 4: $n = 3000$

IMPORTANT NOTE:

$$0 \leq W_i \leq n$$

DOES IMPLY

$$0 \leq M \leq 2n$$

(will be explained)

Solution 3* – Subtask 3 only

PSUEDOCODE

```
INPUT W1, W2, W3
FOR M: 1 - 200
 FOR A: 0 - M-1
  FOR B: 0 - M-1
   IF(W1==(A+B)%M &&
W2==(2A+B)%M &&
W3==(3A+B)%M)
    PRINT A, B, M
    END
PRINT "No solution"
```

Solution 4* – Subtasks 3 and 4 only

PSUEDOCODE

```
INPUT W1, W2, W3
FOR M: 1 - 6000
 FOR A: 0 - M-1
  B = (W1 - A + M) % M
  IF(W2==(2A+B)%M &&
W3==(3A+B)%M)
   PRINT A, B, M
   END
PRINT "No solution"
```

Solution 3* – Subtask 3 only

Subtask	Score	Max Score
1	0	4
2	0	12
3	20	20
4	0	28
5	0	36
Total	**20**	**100**

Solution 4* – Subtasks 3 and 4 only

Subtask	Score	Max Score
1	0	4
2	0	12
3	20	20
4	28	28
5	0	36
Total	**48**	**100**

	$W_1 \leq W_2$	$W_1 \geq W_2$
$W_2 \leq W_3$	Done Subtask 2	**Pending**
$W_2 \geq W_3$	**Pending**	Done Subtask 2

Recall: (From Subtask 2)

Define

$$d_1 = W_2 - W_1$$
$$d_2 = W_3 - W_2$$

KEY: $d_1 \equiv d_2 \equiv A$

$$0 \leq W_i < M$$
$$\Rightarrow -M < d_1, d_2 < M$$

Full Solution

$$d_1 \equiv d_2 \equiv A$$
$$\Rightarrow d_1 - d_2 = kM$$

$$0 \leq W_i < M$$
$$\Rightarrow -M < d_1, d_2 < M$$
$$\Rightarrow -2M < d_1 - d_2 < 2M$$
$$-2M < kM < 2M$$
$$-2 < k < 2$$
$$k = 0 \; or \; k = \pm 1$$

$d_1 - d_2 = 0$ (solved in subtask 2) or $|d_1 - d_2| = M$

Full Solution

$$d_1 - d_2 = 0 \text{ (solved in subtask 2) } or \ |d_1 - d_2| = M$$

- In subtask 2, there are infinitely many solutions for M, and we choose the minimum M by $\max(W_i) + 1$
- KEY: For the latter case, M is UNIQUELY defined as $|d_1 - d_2|$.

Full Solution

Sidetrack: explanation for subtasks 3 and 4

$$0 \leq W_i \leq n$$
$$\Rightarrow -n \leq d_1, d_2 \leq n$$
$$\Rightarrow -2n \leq d_1 - d_2 \leq 2n$$
$$\Rightarrow 0 \leq M = |d_1 - d_2| \leq 2n \ (if \ d_1 \neq d_2)$$

Full Solution

- So the solution is simple:
- If $d_1 = d_2$, $M = \max(W_i) + 1$
- Else $M = |d_1 - d_2|$, and **CHECK** that $W_i < M$
  - No $\Rightarrow$ $No\ solution$
  - The case where d_1 and d_2 have the same sign and $d_1 \neq d_2$ are also treated as "No"
- The rest is the same as subtask 2:
- $A = (d_1 + M)\ \%\ M$
- $B = (W_1 - A + M)\ \%\ M$