Recursion, Divide and Conquer

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Outline of Lecture

1. Function & Procedure
2. Recursion
3. Exhaustion
4. Divide & Conquer

Function

- Function in Maths : Input -> Process -> Output
- e.g. F(x) = x^2 + 2x + 1
- x is the input, F() is the process, y is the Output

- Function in OI :
- Similar meaning in Math
- A program segment that perform : (Input) -> Process -> Output

Function in OI

```
int f(int x) {
    int y = x * x + 2x + 1;
    return y;
}

int main () {
                int val = f(5);
                cout << val;
                return 0;
}
```

Define a function (function name & type/no. of input)

Process of the function

Output of the function

Call the function with parameter x = 5

Function

```
int f(int x) {
        int y = x * x + 2x + 1;
        return y;
}

int main () {
        printf("%d\n", f(3));
        return 0;
}
```

```
int main () {
        y = 3 * 3 + 2 * 3 + 1;
        printf("%d\n", y);
        return 0;
}
```

This two program yield the same output

Function

Similar to Math, there can be more than 1 input value in function
e.g.

```
double dist(double x, double y, double x2, double y2)
{
        double distance = (x - x2) * (x - x2) + (y - y2) * (y - y2);
        return sqrt(distance);
}
```

Procedure

Procedure : A sub-program

Similar to a function but does not give an output

i.e. Input -> Process

Procedure

```
void read() {
        cin >> a >> b;
}

void solve() {
        c = a + b;
        cout << c;
}
int main () {
        read();
        solve();
        return 0;
}
```

```
int main () {
        cin >> a >> b;
        cout << a + b;
        return 0;
}
```

These two program are same

Function & Procedure

Use of Function & Procedure:

1. Make your code cleaner & more understandable
2. Perform recursion

Recursion

Recursion is a function or procedure call itself

e.g.

```
int f(int x) {
        return f(x - 1) + f(x - 2);
}
```

The function f calls itself in the function (f(x) calls f(x-1), f(x-2)), this is a recursion

Recursion

Consider the following program:

```
void f(int x) {
        ans += x * x;
        if (x > 0) f(x - 1);
}


int main () {
        ans = 0;
        f(3);
        cout << ans;
}
```

Recursion

```
void f(int x)  {
        ans += x * x;          // line 1
        if (x > 1) f(x - 1);   // line 2
        ans += ans;            // line 3
}

int main () {
        ans = 0;
        f(3);
        cout << ans;
}
```

The program run in the following order:

1. Call f(3)
2. Run line 1 of f(3)
3. Run line 2 of f(3) -> call f(2)
4. Run line 1 of f(2)
5. Run line 2 of f(2) -> call f(1)
6. Run line 1 of f(1)
7. Run line 2 of f(1) -> won't call f(0) as not (x > 1)
8. Run line 3 of f(1) -> end of f(1)
9. Run line 3 of f(2) -> end of f(2)
10. Run line 3 of f(3) -> end of f(3)

Program does not run sequentially with recursion.

Recursion

With recursion :
Our program can run in a way that is not sequentially
We can “jump” to run the first statement of the sub-program at anytime

Therefore, recursion help us to solve problem with following properties:

1. The problem can divide into same problem with smaller parameter (I call this sub-problem)
2. We need information of sub-problem to solve the current problem

The above problem-solving-method call divide and conquer

We use examples to explore recursion & divide and conquer now

1. Factorial

Problem statement : Find factoral of x (x! = x * (x - 1) * ... 1)
Instead of use for loop to solve, we can use recursion to solve it.

Analysis : x! = x * (x - 1)! where (x - 1)! is same problem with smaller parameter

```
int factorial(int x) {
        return x * factorial(x - 1);
}
```

done.....(?)

1. Factorial

```
int factorial(int x) {
        return x * factorial(x - 1);
}
```

According to this program segment:

f(2) = 2 * factorial(1)
f(1) = 1 * factorial(0)
f(0) = 0 * factorial(-1) ...

Seems I miss something

1. Factorial

We should also tell our program when to stop recurring

```
int factorial(int x) {
        if (x > 0) return x * factorial(x - 1);
        else if (x == 0) return 1;          // 0! = 1
}
```

We call the critiria of stop recursion the base case : if (x == 0) return 1;
We call the relation of problem & sub-problem the recurrance relation : x * factorial(x - 1);

2. Fibonacci number

Problem : Find the x-th fibonacci number

Analysis : f(x) = f(x - 1) + f(x - 2) where f(x - 1) and f(x - 2) is same problem with smaller parameter

```
int f(int x) {
        if (x > 2) return f(x - 1) + f(x - 2);   // recurrance relation
        else if (x == 1 || x == 2) return 1;   // base caseS
}
```

3. Knapsack problem

1. There are N objects, each of them has its value v[i] and weight w[i] (in kg)
2. We can choose buying some of the objects
3. But the total weight of the things we buy cannot larger than W
4. We hope to maximize the total value of the things we buy

Sample input:
4 10.5 (n, W)
7.5 100 (w[i], v[i])
3.1 10
5 50
5.5 51

Sample output: 101 (Buy object 3, 4)

3. Knapsack problem

Greedy approach cannot give us the optimal answer

We can try all combination of find the optimal answer

All combination = {Y, Y, Y, Y}, {Y, Y, Y, N}, {Y, Y, N, Y}, {Y, Y, N, N} ..... {N, N, N, N}

How to code?

3. Knapsack problem

Code :

```
if (n == 1) for (bool a = false; a <= true; a++)
if (n == 2) {
                for (bool a = false; a <= true; a++)
                for (bool b = false; b <= true; b++)
}
if (n == 3) {
                for (bool a = false; a <= true; a++)
                for (bool b = false; b <= true; b++)
                for (bool c = false; c <= true; c++)
}
.....
```

3. Knapsack problem

Think about recursion

Origin problem: Decide whether to buy object 1..N
Assume we buy the first object

New problem: Decide whether to buy object 2..N (same problem with smaller parameter)
Assume we do not buy the first object

New problem: Decide whether to buy object 2..N (same problem with smaller parameter)

We get the recurrence relation!!

3. Knapsack problem

```
void knapsack(int x) {                  //   decide whether to buy object x to N
                // assume we buy object x
                curweight += w[x];
                curvalue += v[x];
                knapsack(x + 1);        // decide whether to buy object x + 1 to N
                // assume we do not buy object x
                curweight -= w[x];
                curvalue -= v[x];
                knapsack(x + 1);
}
```

Don’t forget the base case

3. Knapsack problem

Base case analysis: When we will stop recursion ?

We stop after we had decided whether to buy all of the N objects

knapsack(1) represent we are deciding whether to buy object 1 to N
knapsack(N) represent we are deciding whether to buy object N to N

i.e. When x = N + 1 represent we had decided whether to buy all N objects

3. Knapsack problem

```
void knapsack(int x) {                  //   decide whether to buy object x to N
        if (x == N + 1) {
                if (curweight <= W && curvalue > ans) ans = curvalue;
        }
        else {
                curweight += w[x];
                curvalue += v[x];
                knapsack(x + 1);        // decide whether to buy object x + 1 to N
                curweight -= w[x];
                curvalue -= v[x];
                knapsack(x + 1);
        }
}
```

Exhaustion

In the previous example, we use recursion to Try All Combination which is also call exhaustion

This algorithm is slow but intuitive and correct all the time

We can get partial score by exhaustion in OI contest all the time.

When we encounter difficult problem, use exhaustion to get some score is a good strategy !!

Summary

We can think the problem in the following way :

1. The problem can divide into same problem with smaller parameter

2. What is the recurrence relation?
   3.1. How to divide it to smaller-parameter-problem
   3.2 How to solve if we get the information of smaller-parameter-problem

3. What is the base case?

4. Use recursion to code it !!

Divide and Conquer

In the examples above, we divide the problem f(x) into f(x - 1).

The parameter of sub-problem reduce by 1

In some case we can divide the problem f(x) into sub-problem with even smaller parameter

4. Big mod

Problem : Given B, P, M. Find (B^P) % M

Naive solution :

```
ans = 1;
for (int i = 0; i < P; i++) { ans *= B; ans %= M; }
cout << ans;
```

Time complexity: O(P)

4. Big mod

In the solution above, we use B^P = B^(P - 1) * B

i.e. f(x) = f(x - 1) * B

We divide f(x) to f(x - 1) every time

A better way is to divide f(x) to f(x / 2) every time !!!

4. Big mod

Analysis : Let f(x) = B^x

Recurrance relation:

if (x is even) f(x) = f(x / 2) * f(x / 2)

if (x is odd) f(x) = f(x / 2) * f(x / 2) * B

How about base case? When we stop recurring?

When x = 1, we are calculating B^1. We can direct return the answer B

4. Big mod

```
int f(int x) {
        if (x == 1) return B % M;
        int tmp = f(x / 2) % M;
        if (x % 2 == 0) return (tmp * tmp) % M;
        else return (tmp * tmp * B) % M;
}
```

4. Big mod

Every time we reduce f(x) to f(x / 2) until x = 1

Let N = 30, we only need to call f(30), f(15), f(7), f(3), f(1) which is around lg(N) call

So, time complexity is lg(N)

By divide & conquer, we sometime can reduce the time complexity if we divide the problem appropriately.

Master Theorem

Instead of using intuition, we have a better way to analysis the Time Complexity of D&C

Master Theorem tells us that: for an algorithm run in time complexity $T(n)$

If T(n) can be written as a recurrence formula in form of $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$

We have: $T(n) = \begin{cases} O(n^d) & if\ d > \log_b a \\ O(n^d log n)\ if\ d = \log_b a \\ O(n^{\log_b a})\ if\ d < \log_b a \end{cases}$

Master Theorem

$T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$

This is a representation of the time complexity of a divide and conquer algorithm

T(n) = Total Time Complexity of the original problem

$aT\left(\frac{n}{b}\right)$ = Time Complexity for the divide parts, where we divide the original problem to *a sub-problem, the parameter of each sub-problem is reduced from n to* $\frac{n}{b}$

$O(n^d)$ = Time Complexity for the conquer part

Master Theorem

```
int f(int x) {
        if (x == 1) return B % M;
        int tmp = f(x / 2) % M;
        if (x % 2 == 0) return (tmp * tmp) % M;
        else return (tmp * tmp * B) % M;
}
```

Analysis:
We divide T(x) to T(x / 2): b = 2
We divide T(x) to ONE sub-problem of T(x / 2) (we call T(x / 2) once): a = 1
For conquer part: we use $tmp \times tmp \ \% \ M$ and $tmp \times tmp \times B \ \% \ M$, which is O(1),
As O(n^0) = O(1), so d = 0

Master Theorem

$$T(n) = \begin{cases} O(n^d) & if\ d > \log_b a \\ O(n^d log n)\ if\ d = \log_b a \\ O(n^{\log_b a})\ if\ d < \log_b a \end{cases}$$

For big mod, we have:

$b = 2, a = 1, d = 0$

$log_b a = \log_2 1 = 0 = d$

Therefore, time complexity for big mod = $O(n^d log n) = O(n^0 log n) = O(log n)$

5. L-piece

Constructive problem can often be solved by Divide and Conquer

As constructive problem with small constraint are usually intuitive and it can often be divided into sub-problem with smaller constraint

Problem : Given a N * N grid with a empty cell. Output a way to use L-piece to cover the whole grid where N is a power of 2

Input :	Output:
4	BBCC
0000	BAAC
0000	DAEo
0001	DDEE
0000	

5. L-piece

When we encounter constructive problem, according to Percy Wong, we should try small example

When N = 2, there are 4 cases

10	01	00	00
00	00	10	01

Just put a L-piece on the non-empty cells, done.

5. L-piece

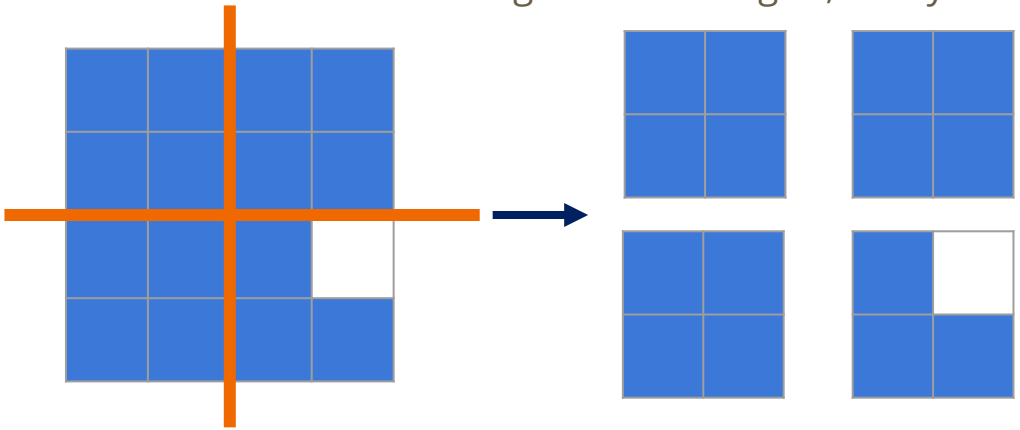
For larger cases (e.g. 4 x 4), can we divide it to same problem with smaller constraint?

Note that in this problem: N = 2 ^ k
So, 3 x 3, 1 x 2... with 1 empty cell is NOT the same problem with smaller constraint as N != 2 ^ k

So, we can just consider can 4 x 4 grid with 1 empty cell divide into 2 x 2 grid with 1 empty cell

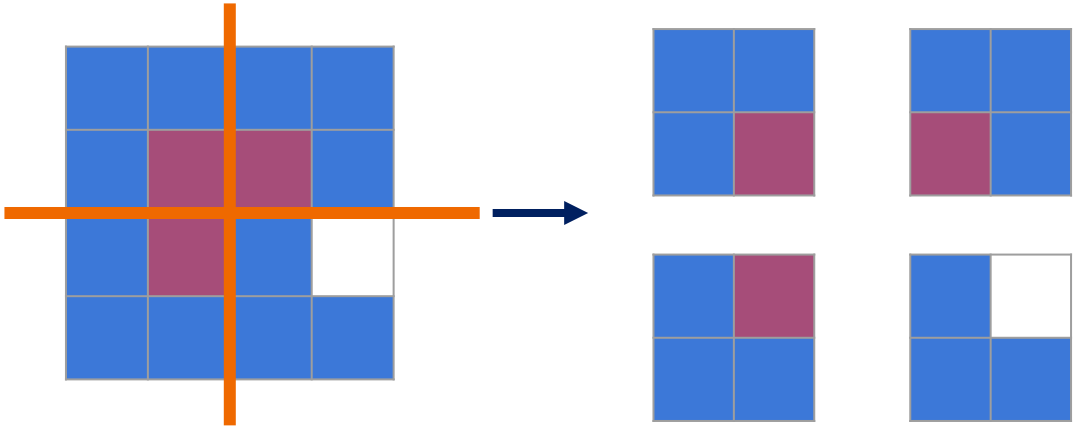
5. L-piece

As we need to divide a 4 x 4 grid to a 2 x 2 grid, let try dividing it into 4 sub-grid



We hope that each sub-grid has exactly 1 empty cell, how can we attain this?

5. L-piece



Yeah!!! We reduce it to smaller-parameter-same-problem!!!

5. L-piece

Final Solution:

1. Divide the grid to 4 sub-grid: upper-right, upper-left, lower-right, lower-left
2. Put a L-piece in a suitable orientation such that each sub-grid has exactly 1 hole
3. Solve the problem-with-smaller-parameter recursively

Time Complexity:

$T(n) = 4\,T(\frac{n}{4}) + O(1)$

$a = b = 4, d = 0$

$T(n) = O\left(n^{\log_4 4}\right) = O(n)$

Code: https://judge.hkoi.org/submission/253773/details?sharing=Xqgyy30QturGXy3xNa8qwlmU

6. Merge sort

Given an array with N integer (N <= 100000)
Output the sorted array

Input :
8
3 4 1 9 8 10 4 5

Output:
1 3 4 4 5 8 9 10

Bubble sort leads to time limit exceeded, we should use faster sorting algorithm
Merge sort can sort an array in O(N lg N) !!!

6. Merge sort

What is merging ?

Given 2 SORTED array, can we combine the 2 array into 1 SORTED array in O(N) time?

e.g.
Array A = {3, 8, 10, 10}
Array B = {2, 3, 7, 8}

We can solve this by pushing the smaller element in the FRONT of 2 array every time

6. Merge sort

Array A	Array B	Resulting Array
3, 8, 10, 10	2, 3, 7, 8	2
3, 8, 10, 10	3, 7, 8	2, 3
8, 10, 10	3, 7, 8	2, 3, 3
8, 10, 10	7, 8	2, 3, 3, 7
8, 10, 10	8	2, 3, 3, 7, 8
10, 10	8	2, 3, 3, 7, 8, 8
10, 10		2, 3, 3, 7, 8, 8, 10
10		2, 3, 3, 7, 8, 8, 10, 10

6. Merge sort

This show that we can merge two sorted array to 1 sorted array in O(|A| + |B|) time

How can we use this skill to sort an array?

Think about divide and conquer !!!

6. Merge sort

Analysis : If we split the sequence to two sequence, and sort each of them, we can merge them in O(|A| + |B|)

Problem : sort the sequence -> sort(1, n)
Divide : We split the sequence to two -> sort(1, n / 2) and sort(n / 2 + 1, n)
Sub-problem : Sort the splitted sequence
Conquer : Merge them in O(|A| + |B|)

6. Merge sort

Problem : sort the sequence
Divide : We split the sequence to two
Sub-problem : Sort the splitted sequence
Conquer : Merge them in O(|A| + |B|)

Assume the procedure sort(1, n) sort the array A from position 1 to n, we can translate the above logic to code.

```
void sort(int start, int end) {
        sort(start, mid);
        sort(mid + 1, end)
        merge(start, mid, mid + 1, end);
}
```

How about base case?

6. Merge sort

Analysis : If the sequence we are sorting have length = 1, we can stop dividing as we need not to sort an array with length = 1

```
void sort(int start, int end) {
        int mid = (start + end) / 2;
        if (start != mid) sort(start, mid);
        if (mid + 1 != end) sort(mid + 1, n);
        merge(start, mid, mid + 1, end);
}
```

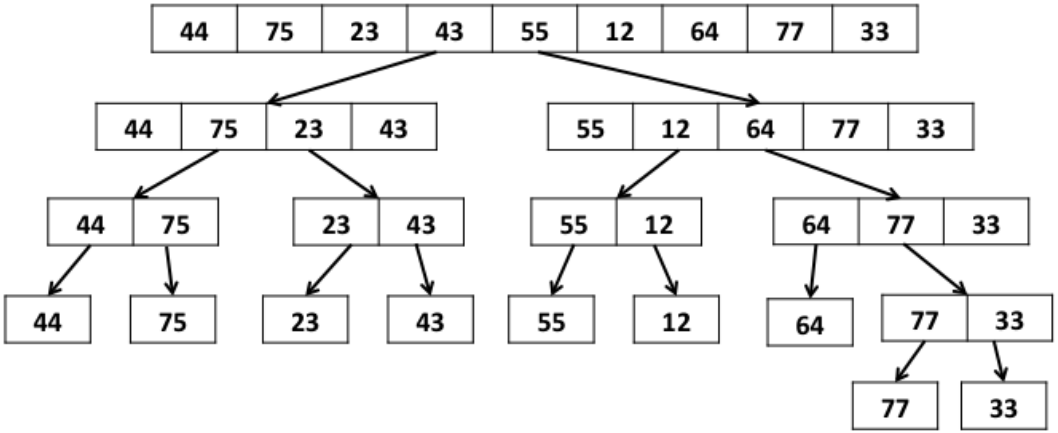
Done.

Implementation of procedure merge:

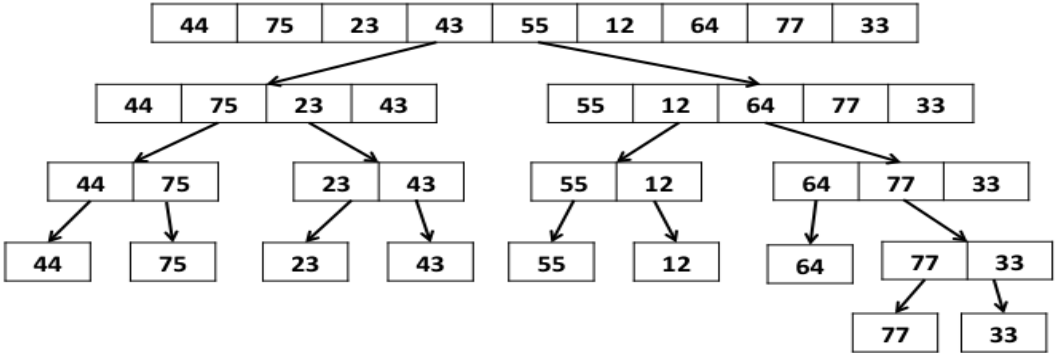
```
void merge(int st1, int ed1, int st2, int ed2) {
   int n = 0, m = 0;
   for(i, st1, ed1) tmpA[n++] = A[i];
   for(i, st2, ed2) tmpB[m++] = A[i];
   int x = 0, y = 0;
   for(i, st, ed2) {
        if (y == m || tmpA[x] <= tmpB[y]) {
             A[i]= tmpA[x];  x++;
        }
        else {
             A[i] = tmpB[y]; y++;
        }
   }
}
```

6. Merge sort

What is the time complexity of quick sort?
Assume our array have the length = 9 which is {44, 75, 23, 43, 55, 12, 64, 77, 33}
We need to divide the sequence like the following



6. Merge sort



Intuitive:

1. The number of element in each layer equal to N (except the last layer)
2. There are at most lg(N) + 1 layer totally
3. We need O(|A| + |B|) = the number of element in the sequence we sort for 1 merging procedure
4. i.e. The time complexity = number of element in the above picture = N * (lg N + 1)= O(N lg N)

Master theorem:

$T(n) = 2\,T(\frac{n}{2}) + O(n)$

$a = b = 2, d = 1$

$T(n) = O(n^1 \log n) = O(nlogn)$

6. Majority (Interactive)

Assume we have a unknown array A[1] ... A[n]
We need to determine if there exists (or which one if exists) an element appear MORE than half
We can ask the following question to the judge: Is A[x] == A[y]?

How many times you need to ask to determine it ?

E.g. We have 4 elements in the array
We ask: A[1] == A[2]? Answer: NO
We ask: A[1] == A[3]? Answer: NO
We ask: A[2] == A[3]? Answer: YES
We ask: A[2] == A[4]? Answer: YES

It can be concluded that A[2] appear more than half

6. Majority (Interactive)

Solution 1:

```
for (int i = 1; i <= n; i++)
{
        same = 0;
        for (int j = 1; j <= n; j++)
                if (ask(i, j) == ‘same’) same++;
        if (same > n / 2) printf(“%d”, i);
}
```

We ask O(n^2) in worst case, although certain optimization (break) can be applied.

6. Majority (Interactive)

Solution 2:

Analysis:
If an element x appear more than half in A[1..n]
It must appear more than half in A[1..n/2] or A[n/2 + 1..n]

Prove: (For simplification, we set n = 16)
x appears less than or equal to half in both A[1..8] and A[9 .. 16]
This means x appears at most 4 times in A[1..8] and A[9..16]
This means x appears at most 8 times in A[1..16]

In general, if x appears no more than half in both A[1..n/2] and A[n/2 + 1..n]
x appears no more than half in A[1..n]

6. Majority (Interactive)

Solution 2: Divide and Conquer

Let f(1, n) = algorithm to find the MORE-THAN-HALF-FREQUENCY number in A[1..n]
We can solve f(1, n / 2) and f(n / 2 + 1, n) first
Obviously, there are at most 1 element appear more than half in a segment

Assume f(1, n / 2) = p, f(n / 2 + 1, n) = q
We can then ask(p, i) and ask(q, i) for all i to determine if they appear more than half in A[1..n]

Number of times we ask: Consider master theorem

$$T(n) = 2\,T\left(\frac{n}{2}\right) + O(n) = O(n \log n)$$

Summary

Be careful what the meaning of SAME Problem with smaller constraint

Sometimes we do not divide the problem f(x) to f(x - 1), but f(x / 2) or other

You should at least familiar with how to use recursion to perform exhaustion (example 3) as it is useful and common

Practise problem:
Exhaustion problem: HKOJ 01031, 01037, 01048, J092, [01049, 01050 (some optimization required)]
Divide and Conquer problem: HKOJ : 01003, 01046, 01047, S134, S163

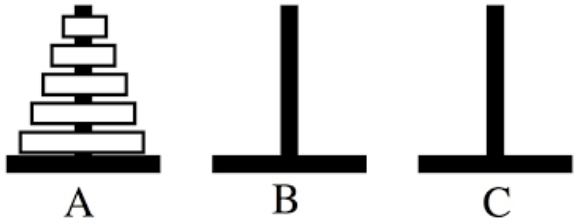
Advance divide & conquer

Some advance d&c technique will (may) be covered in HK-team training.

1. Divide and conquer on tree
2. DP optimized by divide and conquer
3. Closest pair

Extra : Tower of Hanoi

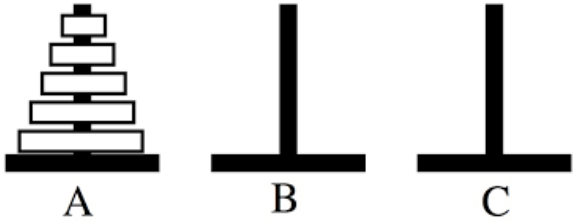
1. We have 3 towers and N pieces of disks
2. We would like to move all N pieces of disks to tower C
3. Each time we should only move 1 disk
4. A larger disk cannot put on the top of a smaller disk



Given n, output a sequence of moves to move all N disks from tower A to C

E.g. N = 2
Output:
Move disk 1 from A to B
Move disk 2 from A to C
Move disk 1 from B to C

Extra : Tower of Hanoi

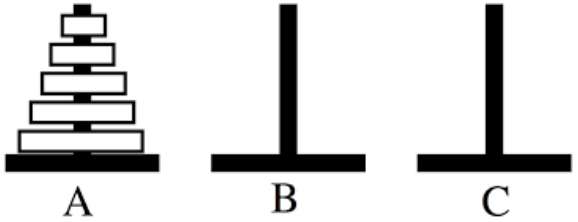


Analysis : If we want to move all N disks from A to C

1. We need to move the Nth (largest) disks from A to C
2. In order to move the Nth disks from A to C, we should move the 1 - N-1 disks from A to B
3. After we move the Nth disks from A to C, our tasks become : move N-1 disks from B to C

Let solve(N, A, B, C) means we need to move ALL N disks from A to C, using B as a temporary storage

Extra : Tower of Hanoi

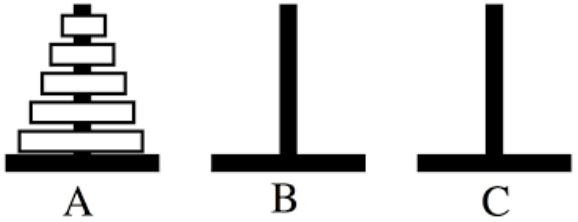


1. We need to move the Nth (largest) disks from A to C
2. In order to move the Nth disks from A to C, we should move the 1 - N-1 disks from A to B
3. After we move the Nth disks from A to C, our tasks become : move N-1 disks from B to C

```
void solve(int n, char A, char B, char C) {
        solve(n - 1, A, C, B);   //   move n-1 disks from A to B
        printf("Move disk %d from %c to %c\n", n, A, C);   // move the Nth disks from A to C
        solve(n - 1, B, A, C)   // move n-1 disks from B to C
}
```

How about base case?

Extra : Tower of Hanoi



Base case analysis : If N = 1, we can directly move it to the destination

```
void solve(int n, char A, char B, char C) {
        if (n == 1) printf("Move disk %d from %c to %c\n", n, A, C);
        else
        {
                solve(n - 1, A, C, B);   //   move n-1 disks from A to B
                printf("Move disk %d from %c to %c\n", n, A, C);   // move the Nth disks from A
to C
                solve(n - 1, B, A, C)   // move n-1 disks from B to C
        }
}
```

Done :)