Data Structures (II)

Lau Chi Yung

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Content

Problem Description

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 1000$
- $1 \leq X \leq 1000$

Input

```
7
insert 39
insert 12
insert 74
exists 39
exists 74
remove 12
exists 12
```

Output

```
yes
yes
no
```

Solution: plain array

`insert` 39

39									

`insert` 12

39	12								

...

`insert` 25

39	12	74	93	62	1	50	25		

Solution: plain array

`insert 39`

39									

`insert 12`

39	12								

...

`insert 25`

39	12	74	93	62	1	50	25		

`remove 12`

39		74	93	62	1	50	25			or
39	74	93	62	1	50	25				or
39	25	74	93	62	1	50				?

Solution: plain array

`insert` 39	39										
`insert` 12	39	12									
				…							
`insert` 25	39	12	74	93	62	1	50	25			
	39		74	93	62	1	50	25			or
`remove` 12	39	74	93	62	1	50	25				or
	39	25	74	93	62	1	50				?

`exists` 74 perform linear search

Solution: plain array

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 1000$
- $1 \leq X \leq 1000$

Time complexity:

`insert`	$O(1)$
`remove`	$O(N)$
`exists`	$O(N)$

For N operations, worst case $O(N^2)$

Solution: sorted array

`insert` 39	39									
`insert` 12	12	39								

...

`insert` 25	1	12	25	39	50	62	74	93		

Solution: sorted array

`insert 39`	39									
`insert 12`	12	39								
			...							
`insert 25`	1	12	25	39	50	62	74	93		
`remove 12`	1	25	39	50	62	74	93			

Solution: sorted array

`insert 39`	39									
`insert 12`	12	39								
			...							
`insert 25`	1	12	25	39	50	62	74	93		
`remove 12`	1	25	39	50	62	74	93			

`exists 74` perform binary search

Solution: sorted array

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 1000$
- $1 \leq X \leq 1000$

Time complexity:

`insert`	$O(N)$
`remove`	$O(N)$
`exists`	$O(\log N)$

For N operations, worst case $O(N^2)$

Solution: hybrid

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 1000$
- $1 \leq X \leq 1000$

Dominant operation	Solution array
`insert`	plain or sorted?
`exists`	plain or sorted?

Solution: hybrid

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 1000$
- $1 \leq X \leq 1000$

Dominant operation	Solution array
`insert`	plain or sorted?
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Solution: hybrid

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 1000$
- $1 \leq X \leq 1000$

Dominant operation	Solution array
`insert`	plain or sorted?
`exists`	plain or sorted?

Solution: hybrid

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 1000$
- $1 \leq X \leq 1000$

Dominant operation	Solution array
`insert`	plain or sorted?
`exists`	plain or sorted?

For N operations, still $O(N^2)$

Problem Description - Level Up

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 1000$

Solution: counting array

Similar idea with counting sort

`insert` 12

											1					
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

`insert` 17

											1					1
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

`insert` 6

					1						1					1
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Solution: counting array

Similar idea with counting sort

`insert` 12

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
											1					

`insert` 17

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
											1					1

`insert` 6

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
					1						1					1

`remove` 12

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
					1											1

Solution: counting array

Similar idea with counting sort

`insert` 12

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
											1					

`insert` 17

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
											1					1

`insert` 6

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
					1						1					1

`remove` 12

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
					1											1

`exists` 12 array lookup

Solution: counting array

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 1000$

Time complexity:

`insert`	$O(1)$
`remove`	$O(1)$
`exists`	$O(1)$

For N operations, worst case $O(N)$

Problem Description - Level Up

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^8$
- Memory limit: 32MB

Problem Description - Level Up

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^8$
- Memory limit: 32MB

Can we use counting array?

```
int a[100000000 + 1];
```

$4 \times 10^8 = 400\text{MB}$ MLE

Problem Description - Level Up

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^8$
- Memory limit: 32MB

"`int`" contains 32 bits

```
int a[3125000 + 1];
```

$4 \times 3125000 = 12.5\text{MB}$

`insert` X	`a[X/32] |= 1<<X%32`
`remove` X	`a[X/32] &= ~(1<<X%32)`
`exists` X	`a[X/32] & 1<<X%32`

Problem Description - Level Up Again

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^{18}$
- Memory limit: 32MB

Problem Description - Level Up Again

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^{18}$
- Memory limit: 32MB

Observation:
There can be at most 10000 distinct values of X

Problem Description - Level Up Again

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^{18}$
- Memory limit: 32MB

Observation:
There can be at most 10000 distinct values of X

Perform *discretization*:

1. Preprocess all X into a sorted array
2. Using binary search, transform the range of X from $[1, 10^{18}]$ to $[1, 10000]$
3. Then use counting array

Problem Description - Level Up Yet Again

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^{18}$
- Memory limit: 32MB
- Preprocess of operations is not allowed (how?)

Problem Description - Level Up Yet Again

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^{18}$
- Memory limit: 32MB
- Preprocess of operations is not allowed (how?)

- Perhaps the task is interactive

Problem Description - Level Up Yet Again

N operations:

- `insert` X
- `remove` X
- `exists` X

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^{18}$
- Memory limit: 32MB
- Preprocess of operations is not allowed (how?)

- Perhaps the task is interactive
- Perhaps each operation's input depends on the output of previous operations

Observation

- We will encounter only 10000 integers in $[1, 10^{18}]$
- Intuitively, X mod 10^8 will *probably* be distinct
- Assume for any $X_i \neq X_j$, $X_i \bmod 10^8 \neq X_j \bmod 10^8$
- We can replace all X with X mod 10^8 and use bitwise counting array!
- How *likely* will our assumption hold?

Birthday problem

Probability that all 23 people have distinct birthdays

$$P = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{343}{365} \approx 0.492703$$

For 70 people, the probablity is 0.000840

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For 70 people, the probablity is 0.000840

Assume X is uniformly distributed over $[1, 10^{18}]$

Then X mod 10^8 is uniformly distributed over $[0, 10^8 - 1]$

Probability that all 10000 X mod 10^8 are distinct

$$P = \frac{10^8}{10^8} \times \cdots \times \frac{10^8 - 9999}{10^8} \approx 0.606551$$

Not too bad.

Birthday problem

$$P = \frac{10^8}{10^8} \times \cdots \times \frac{10^8 - 9999}{10^8} \approx 0.606551$$

Had we used 2×10^8 (which amounts to 25MB in bitwise counting array),

$$P = \frac{2 \times 10^8}{2 \times 10^8} \times \cdots \times \frac{2 \times 10^8 - 9999}{2 \times 10^8} \approx 0.778817$$

Intuitively, with more buckets available, less likely will collision occur.

This does not work well for batched testcases.

Probablility that we get all 5 testcases correct in a batch $= 0.778817^5 = 0.286534$

Collision Handling 1

Rather than giving up on collisions, we try to handle them.

Want to map X into X mod 10.

`insert` X: put X in the *nearest* empty bucket after X mod 10

	0	1	2	3	4	5	6	7	8	9
`insert` 12			12							
`insert` 34			12		34					
`insert` 22			12	22	34					
`insert` 42			12	22	34	42				

Collision Handling 1

`remove` X: search for the bucket containing X, replace it with -1

		12	22	34	42				
0	1	2	3	4	5	6	7	8	9

`remove` 22

		12	-1	34	42				
0	1	2	3	4	5	6	7	8	9

`exists` X: scan from bucket X mod 10 until empty bucket

Collision Handling 1

`insert` X $O(1)$ average

`remove` X $O(1)$ average

`exists` X $O(1)$ average

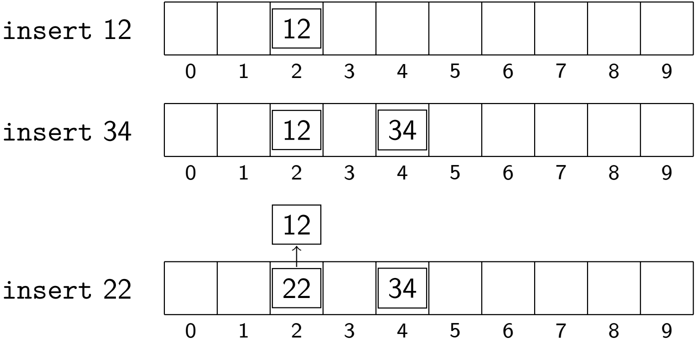
This kind of collision handling is called *Open Addressing*

Variants

- Linear probing
- Quadratic probing

Collision Handling 2

Store all collisions in linked lists.



Collision Handling 2

`insert` X	add item to the linked list at bucket X mod 10 $O(1)$ average
`remove` X	remove item in the linked list at bucket X mod 10 $O(1)$ average
`exists` X	scan the linked list at bucket X mod 10 $O(1)$ average

This kind of collision handling is called *Separate Chaining*

Hash Table

The data structure that we built is called *Hash Table*.

The core function that we used

$$h(X) = X \bmod P$$

is called the *hash function*, which maps *something* into a range of integers that we can handle.

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The core function that we used

$$h(X) = X \bmod P$$

is called the *hash function*, which maps *something* into a range of integers that we can handle.

Is it better to use a prime number for P?

Hash Table

The performance of a hash table is determined by the load factor $\frac{N}{K}$, where N is the number of items and K is the number of buckets.

For *Separate Chaining*, length of linked lists is proportional to load factor

Load factor	Wasted space	Performance
smaller	smaller or larger?	faster or slower?
larger	smaller or larger?	faster or slower?

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C++ Standard Template Library

- C++11 only

```cpp
#include <unordered_set>
using namespace std;
...
unordered_set<int> s;
s.insert(123);
int x = s.count(123); //1
s.erase(123);
```

```cpp
#include <unordered_map>
using namespace std;
...
unordered_map<int, int> m;
m[123] = 456;
int x = m[123]; //456
m.erase(123);
```

Problem Description

N operations:

- `insert` X
- `get min`
  (print the minimum item)
- `remove min`
  (remove the minimum item)

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^{18}$

Input

```
6
insert 12
insert 74
get min
insert 39
remove min
get min
```

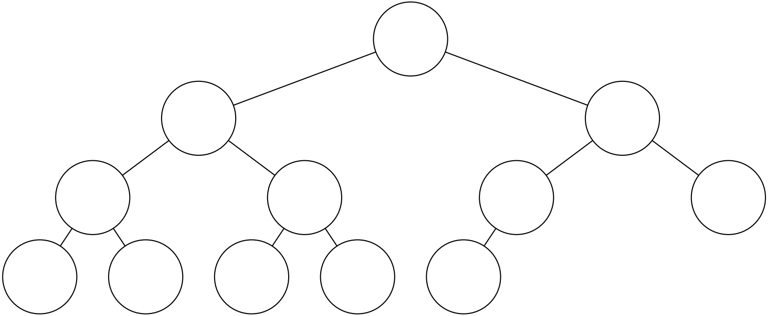
Output

```
12
39
```

Binary Heap

Shape property: compelete binary tree

- The bottom level is filled from left to right
- All above levels are fully filled



Binary Heap

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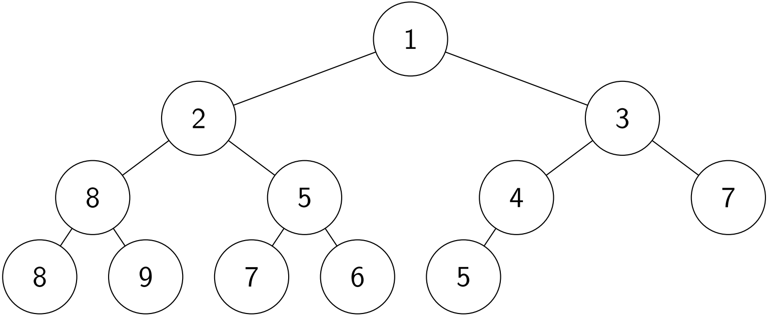
Heap property: every node $\geq$ its parent



Binary Heap

Shape property: compelete binary tree

- The bottom level is filled from left to right
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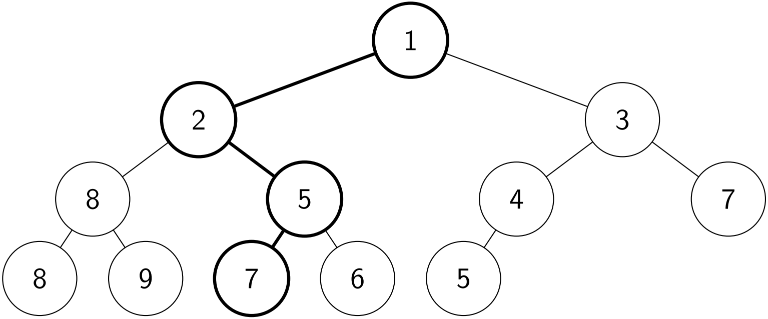
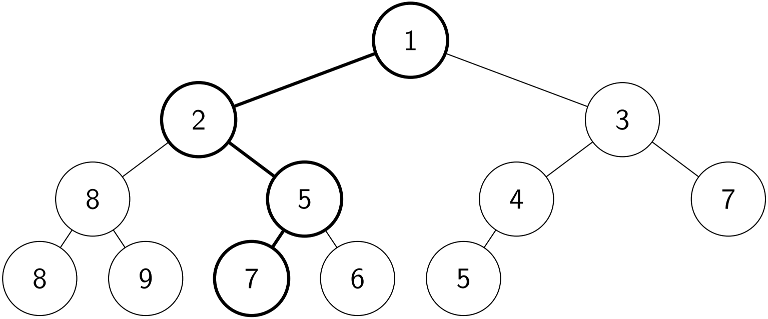
Heap property: every node $\geq$ its parent

Are all root-to-leaf paths sorted?



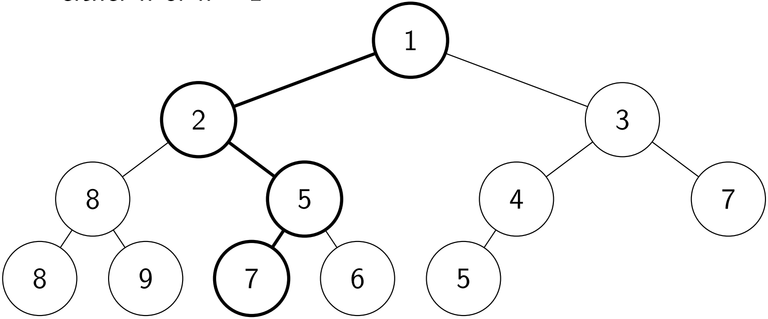
Binary Heap

- Number of nodes $= N$
- Tree height $h =$ maximum number of edges from root to leaf
- $2^h \leq N \leq 2^{h+1} - 1$
- $h \leq \log_2 N = O(\log N)$
- What are the possible lengths of all root-to-leaf paths?



Binary Heap

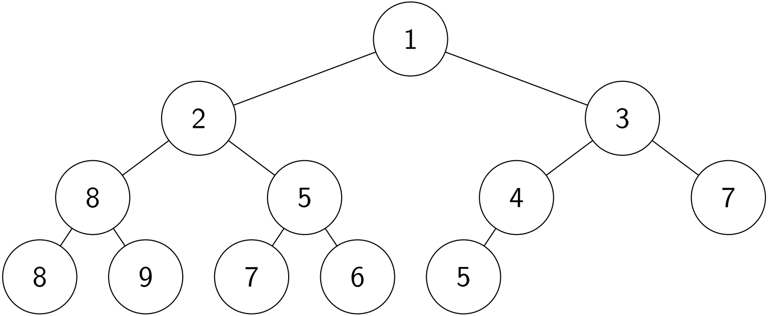
- Number of nodes $= N$
- Tree height $h =$ maximum number of edges from root to leaf
- $2^h \leq N \leq 2^{h+1} - 1$
- $h \leq \log_2 N = O(\log N)$
- What are the possible lengths of all root-to-leaf paths?
- either h or $h - 1$



Binary Heap

`insert` 2

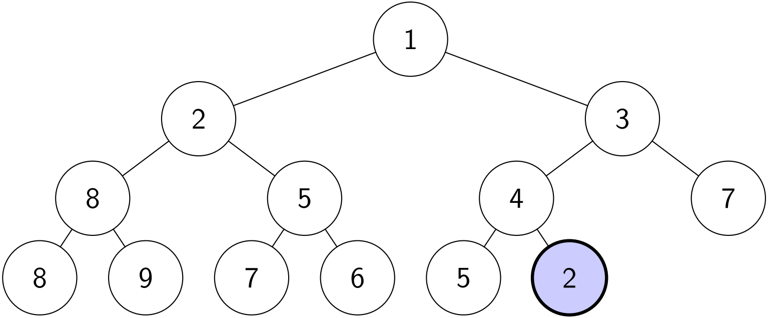
- Add new node at the bottom



Binary Heap

`insert` 2

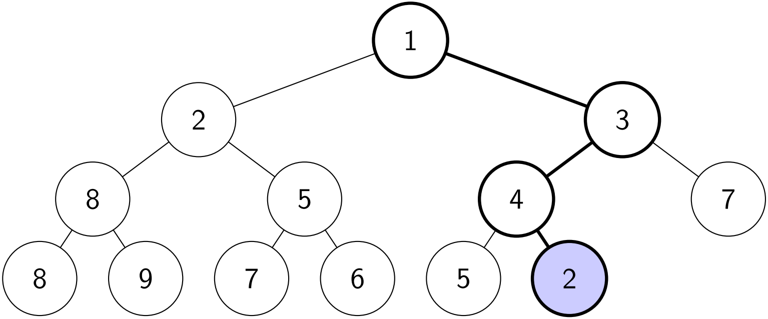
- Add new node at the bottom
- Heap property violated
- How many root-to-leaf paths are not sorted?



Binary Heap

`insert` 2

- Add new node at the bottom
- Heap property violated
- How many root-to-leaf paths are not sorted?
- Only one, let's (merge/quick/bubble/insertion/selection) sort that path



Binary Heap

`insert` 2

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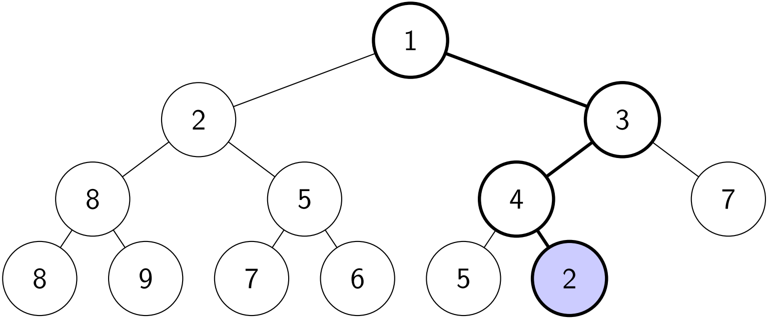
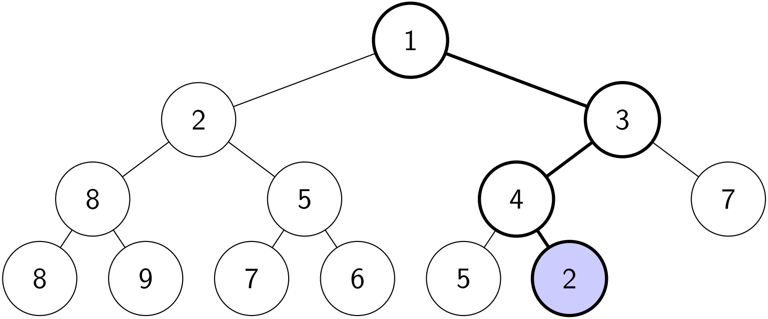


Binary Heap

`insert` 2

- Add new node at the bottom
- Heap property violated
- How many root-to-leaf paths are not sorted?
- We call it *sift-up*



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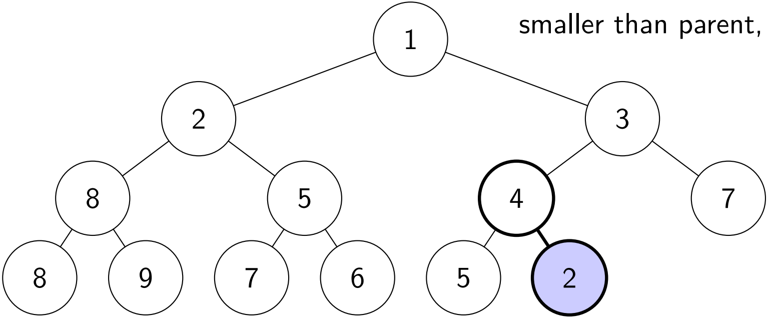

Binary Heap

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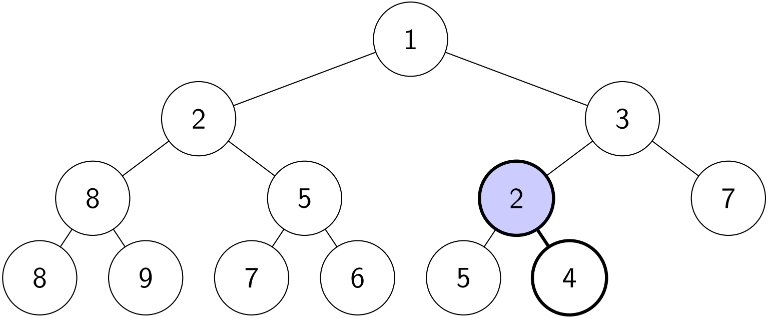

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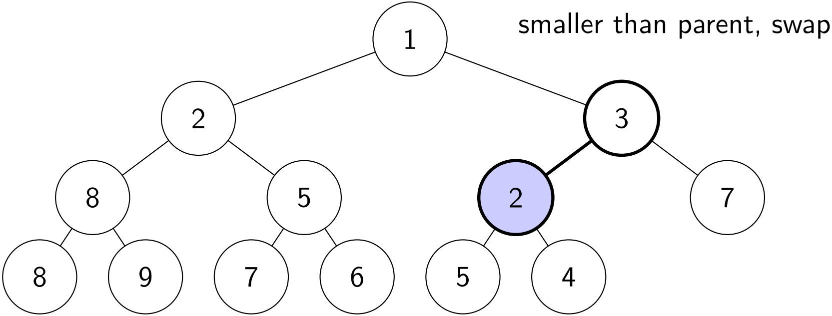

Binary Heap

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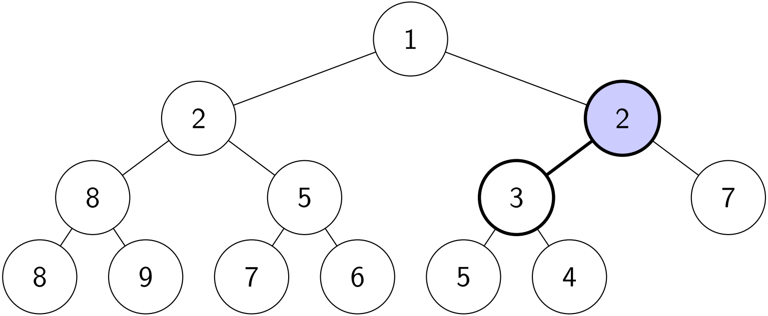

Binary Heap

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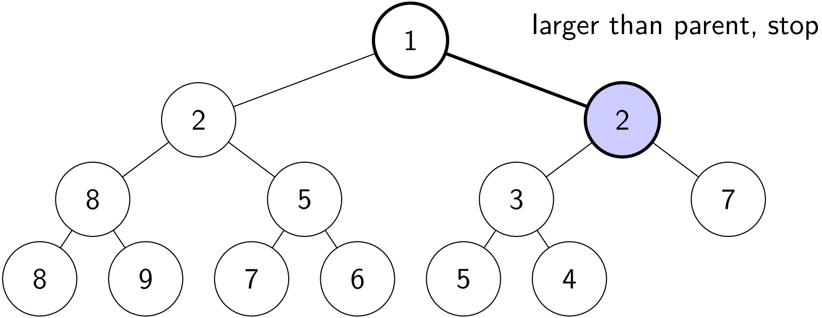
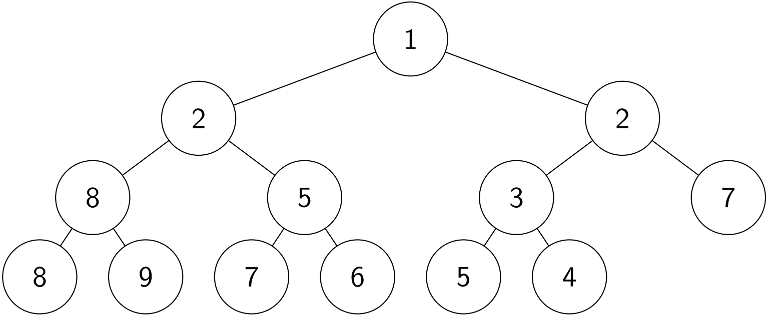


Binary Heap

`insert` 2

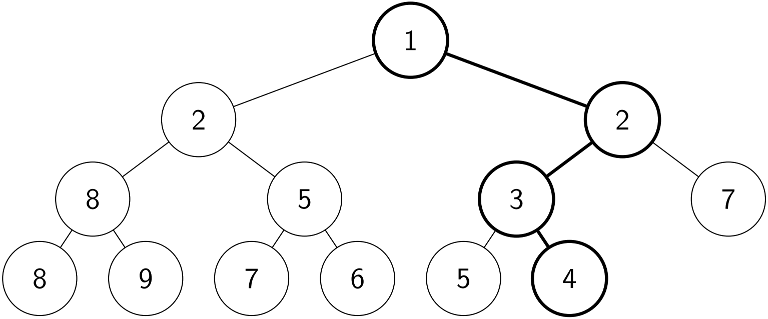
- Add new node at the bottom
- Heap property violated
- How many root-to-leaf paths are not sorted?
- We call it *sift-up*
- Done?



Binary Heap

Proof

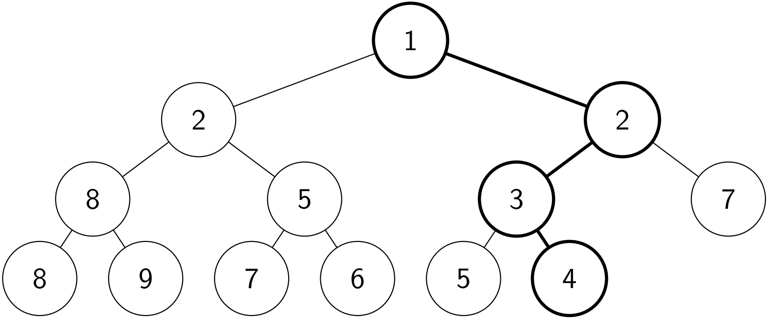
- The path on which we performed sift-up must be sorted
- Sift-up modified the *prefix*es of some other root-to-leaf paths
- Those *prefix*es must be sorted
- Those *prefix*es could only become smaller after sift-up
- So, all root-to-leaf paths are sorted



Binary Heap

Time complexity

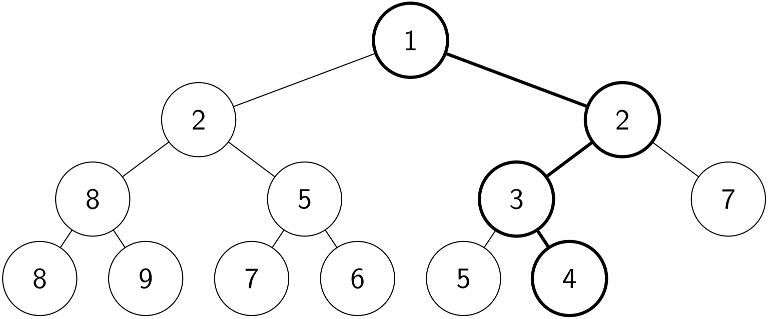
- Maximum number of swaps $= h = O(\log N)$



Binary Heap

`get min`

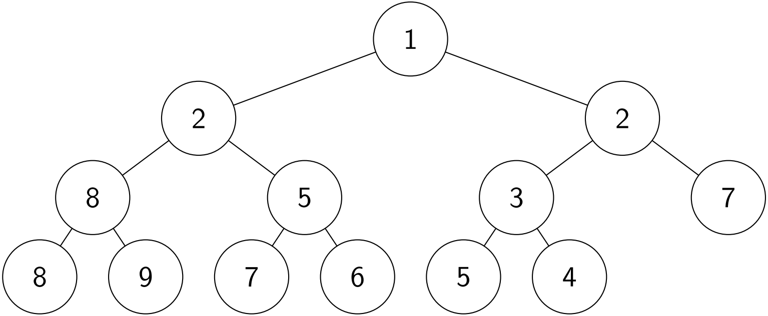
- Simply return root
- $O(1)$



Binary Heap

`remove min`

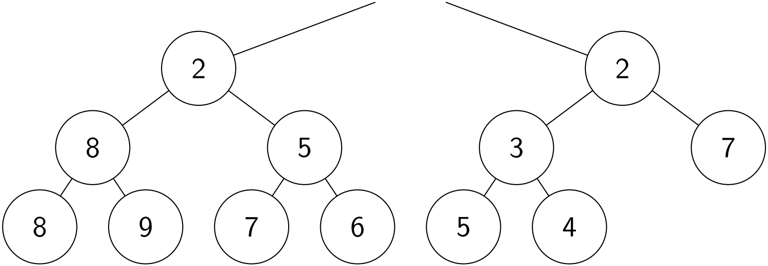
- Just remove it?



Binary Heap

`remove min`

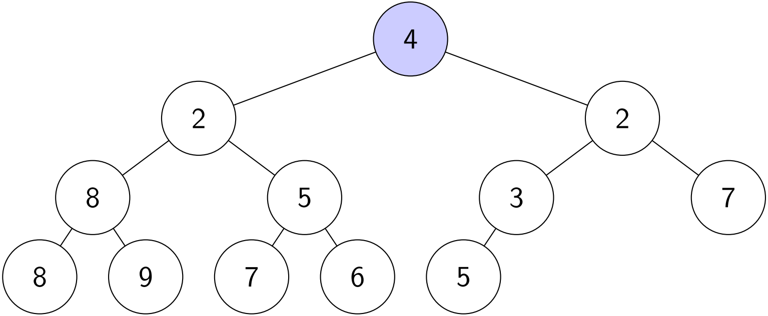
- Just remove it?
- Shape property violated
- Let's replace it with the bottom node



Binary Heap

`remove min`

- Heap property violated
- How many root-to-leaf paths are not sorted?



Binary Heap

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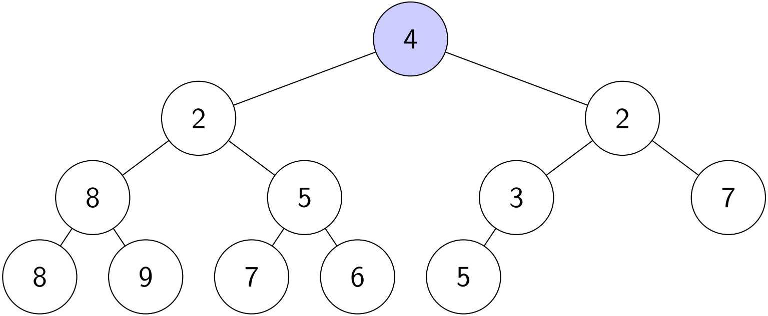
- Perform *sift-down*



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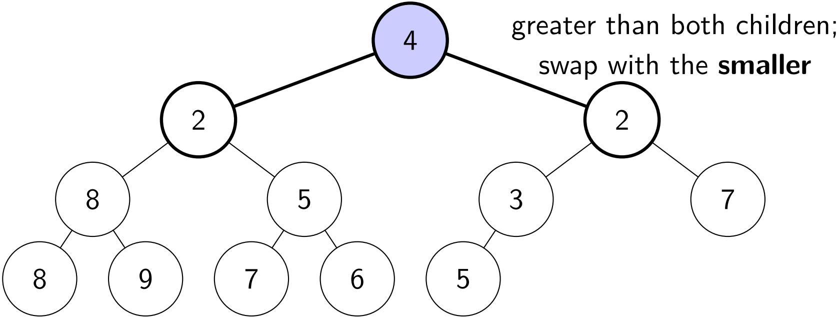

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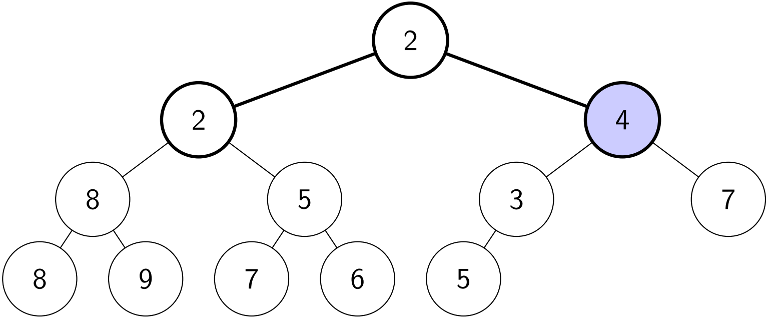

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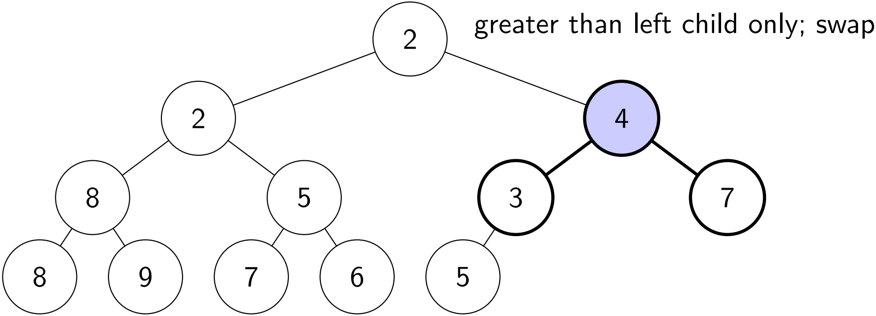

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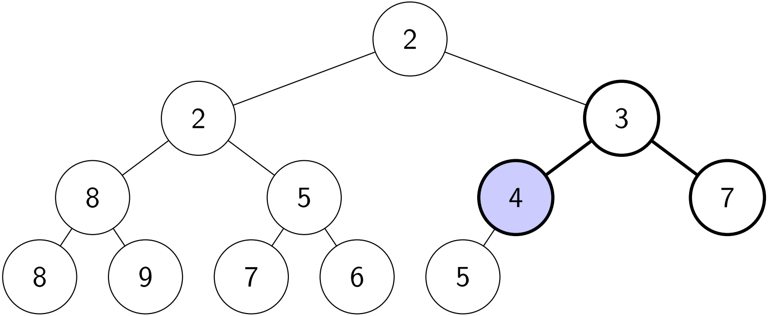

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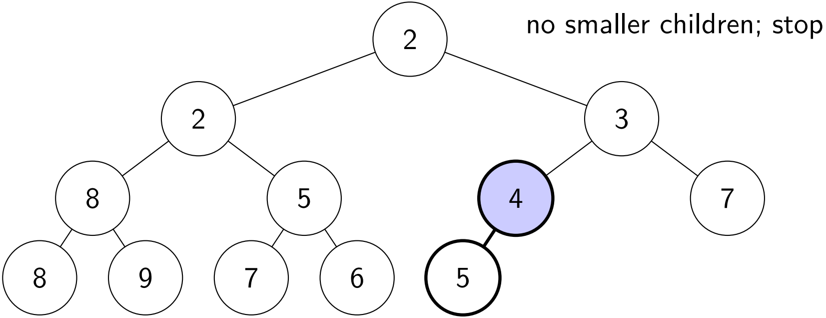
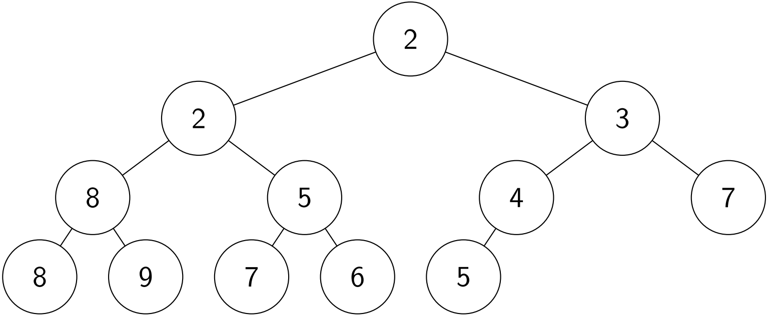


Binary Heap

`remove min`

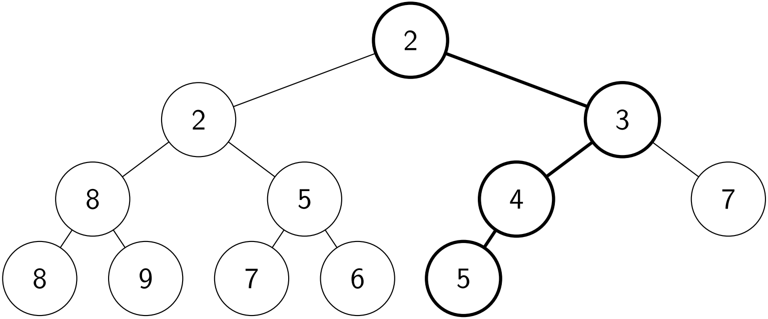
- Heap property violated
- How many root-to-leaf paths are not sorted?
- Perform *sift-down*
- Done?



Binary Heap

Proof

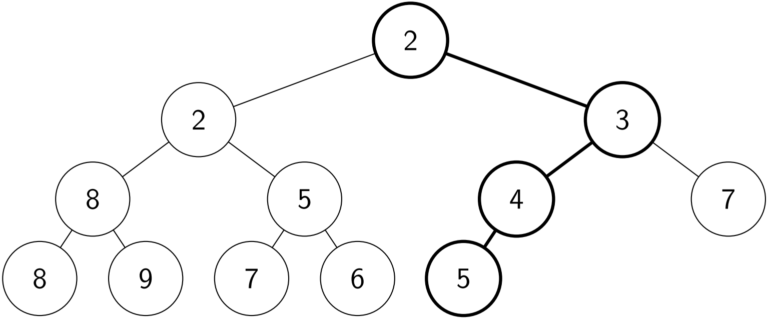
- The path on which we performed sift-down must be sorted
- Sift-down modified the *prefix*es of some root-to-leaf paths
- Those *prefix*es must be sorted
- Those *prefix*es could not become larger than the postfixes
- So, all root-to-leaf paths are sorted



Binary Heap

Time complexity

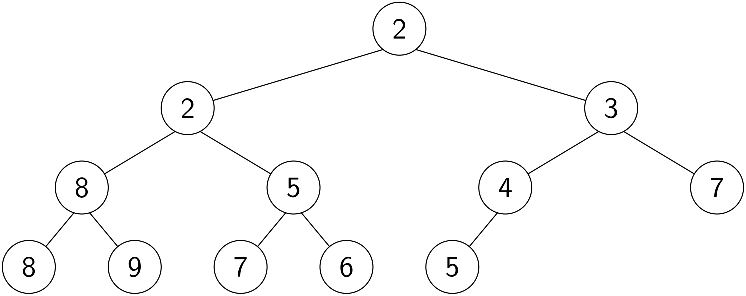
- Maximum number of swaps $= h = O(\log N)$



Binary Heap

Implementation

- plain array



Binary Heap

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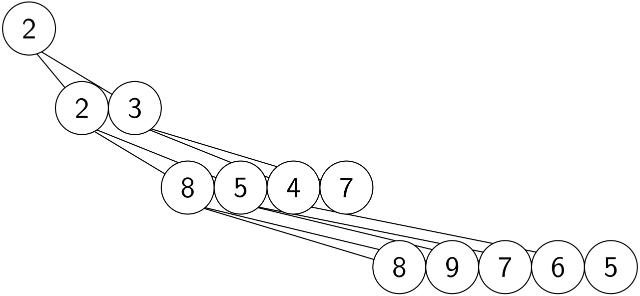
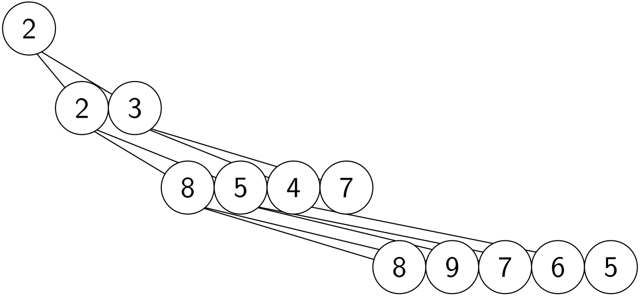


Binary Heap

Implementation

- plain array

- parent i
- left child $2i$
- right child $2i + 1$



	2	2	3	8	5	4	7	8	9	7	6	5
0	1	2	3	4	5	6	7	8	9	10	11	12

Heapsort

1. Build heap naively: $O(N \log N)$
2. Repeatedly remove min: $O(N \log N)$

Any similarity between heapsort and selection sort?

Heapsort

1. Build heap naively: $O(N \log N)$
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Any similarity between heapsort and selection sort?

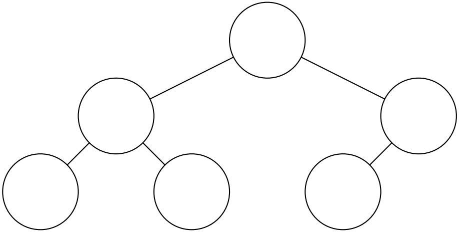
Heapsort vs Quicksort

- In reality, the computer reads not only one memory slot at a time, but a chunk of contiguous memory into cache
- Heapsort does not operate on contiguous memory, introducing cache miss
- Quicksort operate on contiguous memory, benefits from cache

Build binary heap in linear time

- Create the shape first
- Add elements in the lowest levels first
- Perform sift-downs

3	2	1	4	6	7



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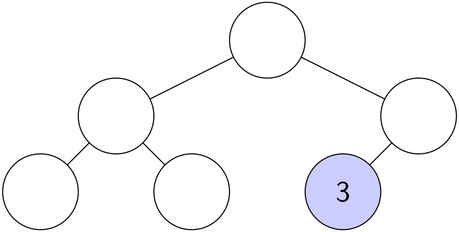

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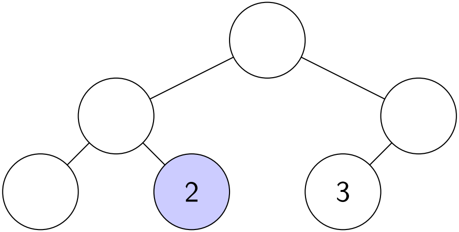

Build binary heap in linear time

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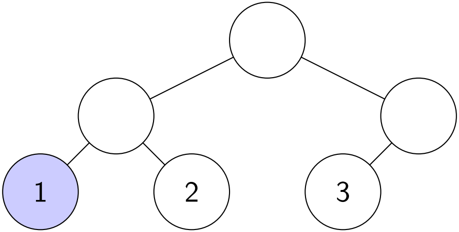
4	6	7



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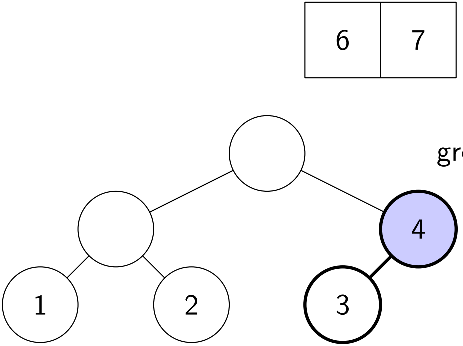

greater than left child; swap

Build binary heap in linear time

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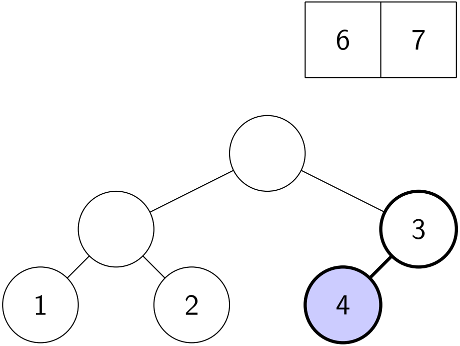

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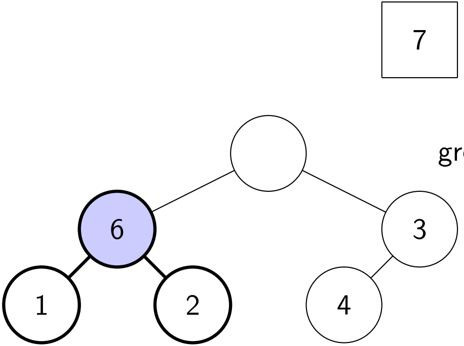

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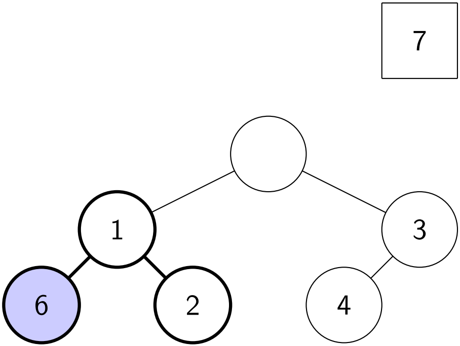

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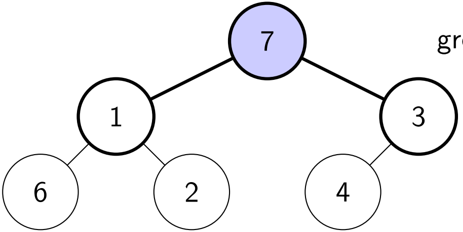

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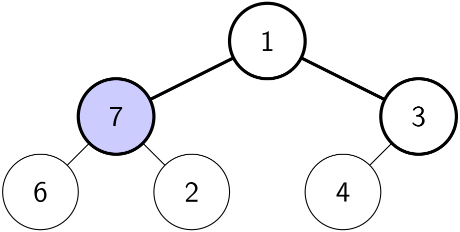

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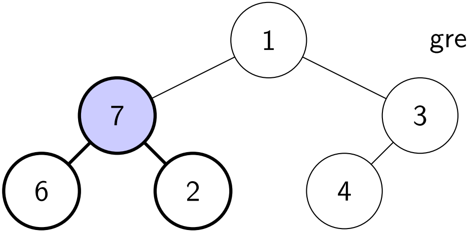

greater than right child; swap

Build binary heap in linear time

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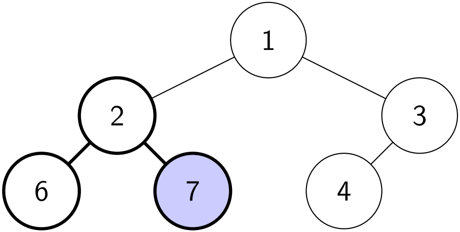

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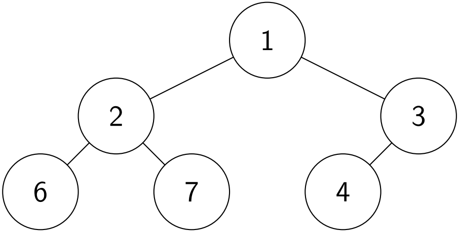
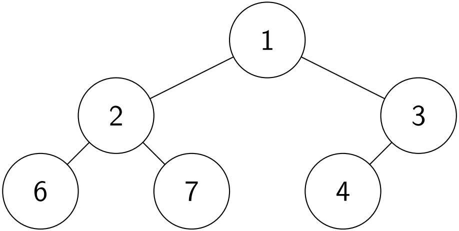


Build binary heap in linear time

	layer	#swaps	#nodes
(top)	1^{st}	h	1
	2^{nd}	$h-1$	2
	3^{rd}	$h-2$	4
		$\dots$	
	h^{th}	1	2^{h-1}
(bottom)	$(h+1)^{th}$	0	2^h



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(bottom)	$(h+1)^{th}$	0	2^h

$$\text{Total \#swaps} = \sum_{i=0}^{h} 2^{h-i} i = 2^h \sum_{i=0}^{h} \frac{i}{2^i} \le 2^h \sum_{i=0}^{\infty} \frac{i}{2^i} = 2^h(2) = O(N)$$

because

$$\sum_{i=0}^{\infty} \frac{i}{2^i} = \sum_{i=0}^{\infty} \frac{i+1}{2^{i+1}} = \sum_{i=0}^{\infty} \frac{i}{2^{i+1}} + \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2}\sum_{i=0}^{\infty} \frac{i}{2^i} + 1$$

$$\Rightarrow \sum_{i=0}^{\infty} \frac{i}{2^i} = 2$$

Build binary heap in linear time

- Time complexity: $O(N)$
- Does that mean we can sort an array in $O(N)$ time?

Build binary heap in linear time

- Time complexity: $O(N)$
- Does that mean we can sort an array in $O(N)$ time?
- No, removing all minimum still need $O(N \log N)$

Merging binary heaps

- Can we merge two binary heaps in $O(N \log N)$ time?

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Merging binary heaps

- Can we merge two binary heaps in $O(N \log N)$ time?
  insert all elements from one to another
- Can we merge two binary heaps in $O(N)$ time?
  concatenate two arrays and build heap
- Can we merge faster?
- Yes, but very complicated
- Better use alternatives such as *Leftist heap* or *Binomial heap*

C++ Standard Template Library

- The afore introduced binary heap was min-heap
- `priority_queue` is max-heap

```
#include <queue>
...
std::priority_queue<int> q;
q.push(4); q.push(7); q.push(3);
int x = q.top();       //x = 7
q.pop(); x = q.top(); //x = 4
```

Problem Description

N operations:

- `insert` X
- `exists` X
- `remove` X
- `get min`

Constraints:

- $1 \leq N \leq 10000$
- $1 \leq X \leq 10^{18}$

Input

```
6
insert 12
insert 39
insert 74
exists 74
remove min
get min
```

Output

```
yes
39
```

Binary Search Tree

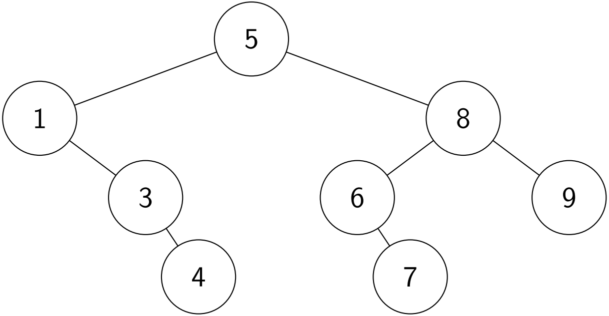
Binary tree

Every node $>$ its left descendants

Every node $<$ its right descendants

Do we get a sorted array in

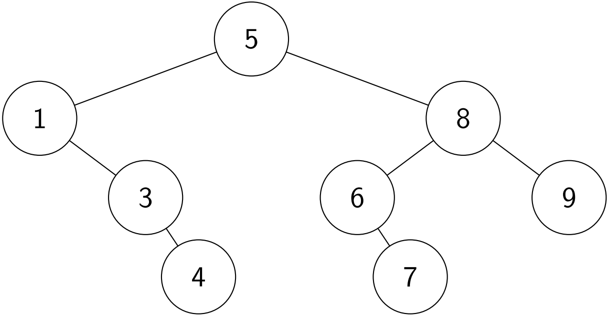
- pre-order traversal?
- in-order traversal?
- post-order traversal?



Binary Search Tree

`exists` 6

- Binary search from root



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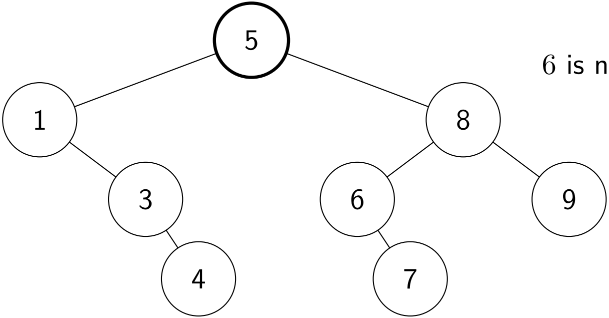

6 is not here; go right

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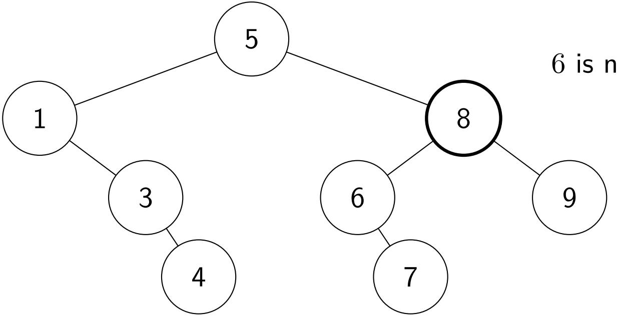

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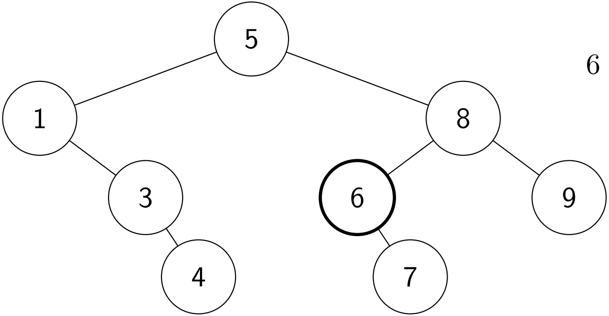
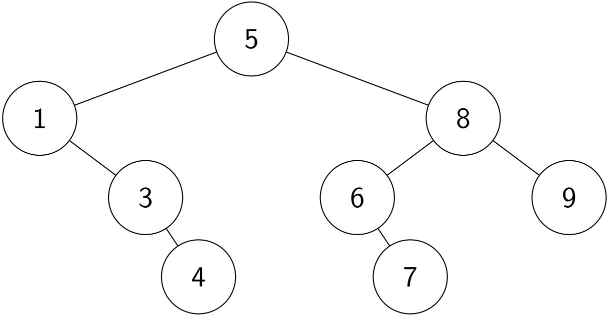


6 is here; done

Binary Search Tree

`exists` 6

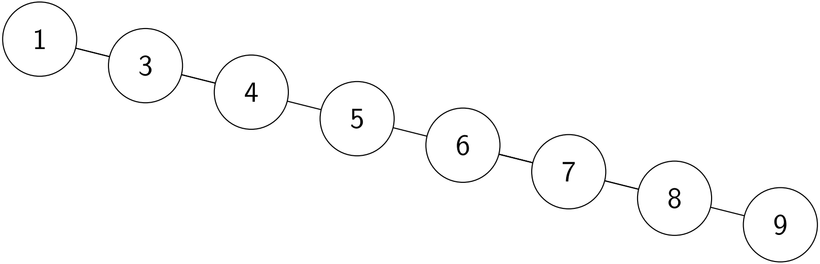
- Binary search from root
- Each iteration we discard half of the subtree
- Time complexity $= O(\log N)$ in average



Binary Search Tree

`exists` 6

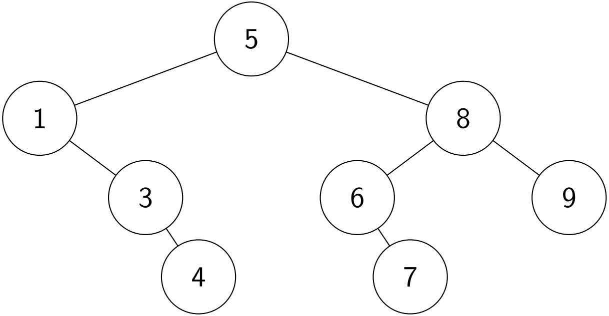
- Binary search from root
- Each iteration we discard half of the subtree
- Time complexity $= O(\log N)$ in average
- Time complexity $= O(N)$ in worst case



Binary Search Tree

`insert` 2

- Binary search a place to insert



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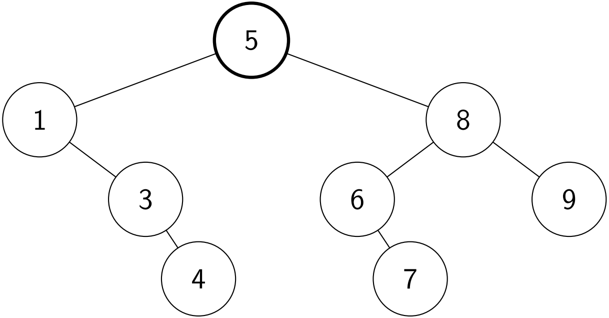

go left

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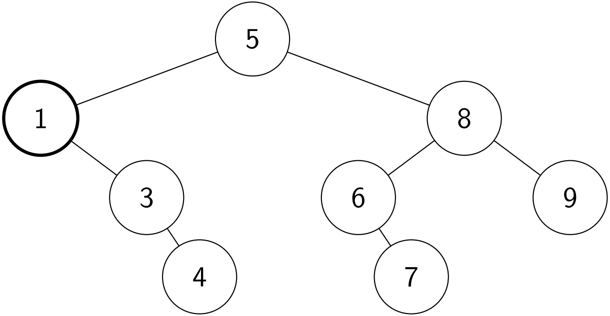

go right

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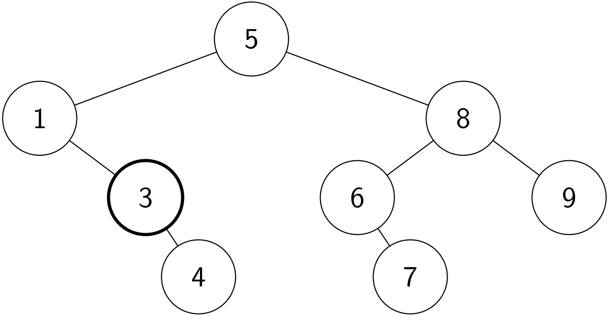

go left

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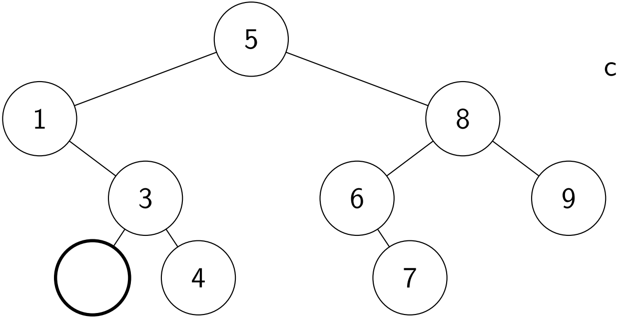
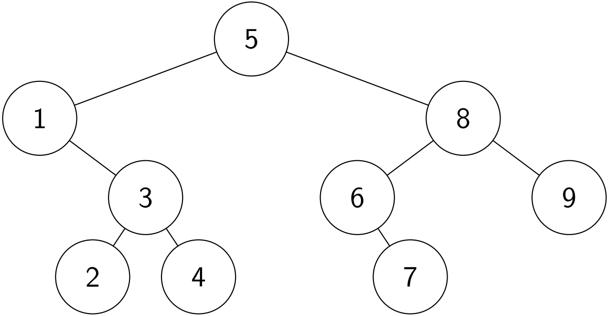


create node

Binary Search Tree

`insert` 2

- Binary search a place to insert
- Each iteration we discard half of the subtree
- Time complexity $= O(\log N)$ in average



insert

Binary Search Tree

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- Time complexity $= O(\log N)$ in average
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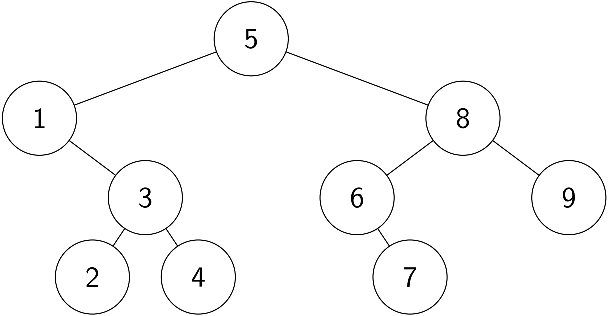
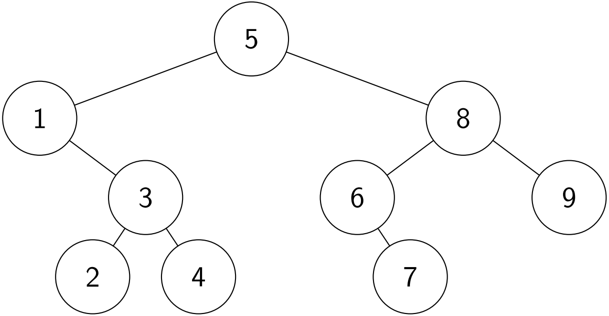


Binary Search Tree

`remove` 8

`remove` 5

- Binary search the node
- Replace it with the greatest node less than it



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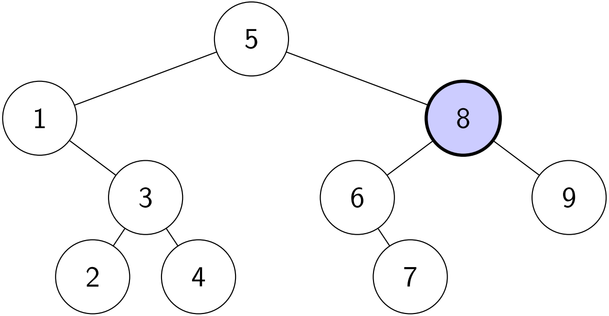

go left

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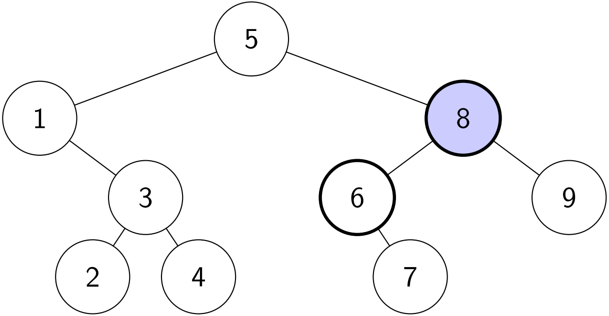

go right

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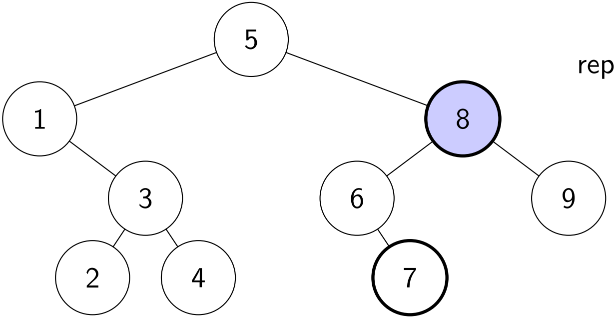

replace 8 with 7

Binary Search Tree

`remove` 8

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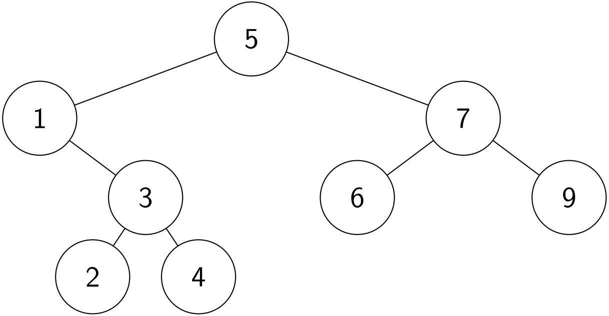

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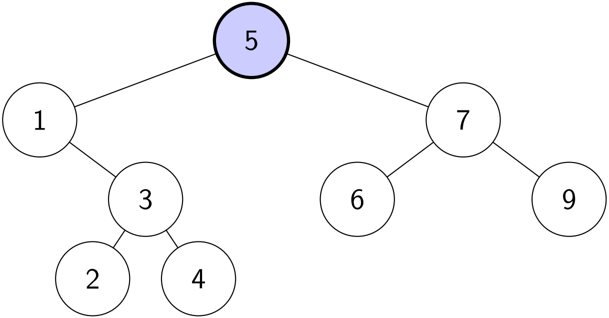

go left

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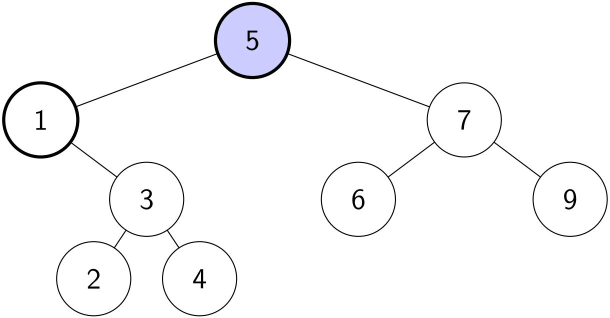

go right

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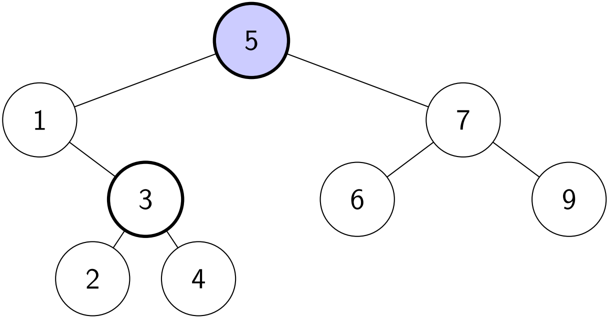

go right

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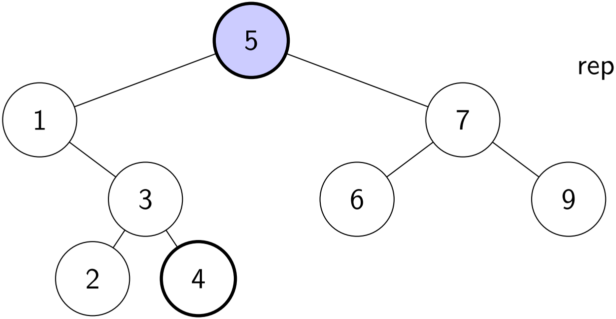
- Binary search the node
- Replace it with the greatest node less than it



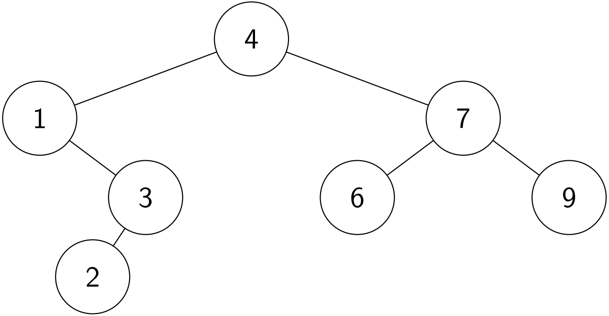
replace 5 with 4

Binary Search Tree

`remove` 8

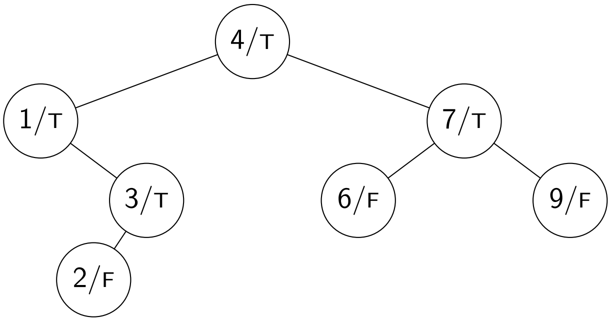
`remove` 5

- Binary search the node
- Replace it with the greatest node less than it
- Locate the node $= O(\log N)$ in average
- Locate the smaller node $= O(\log N)$ in average
- Time complexity $= O(\log N)$ in average
- Time complexity $= O(N)$ in worst case



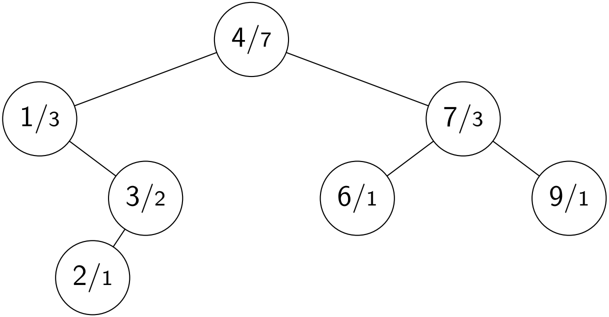
Augmented Binary Search Tree

- We can attach *value*s to nodes, e.g.
  - existence
- Nodes are sorted by their *key*s
- How does it affect `insert` and `remove`?
- Do we still need `remove`?



Augmented Binary Search Tree

- We can attach *values* to nodes, e.g.
  - existence
  - subtree size
- Nodes are sorted by their *keys*
- How does it affect `insert` and `remove`?
- Can we find the k^{th} largest element?

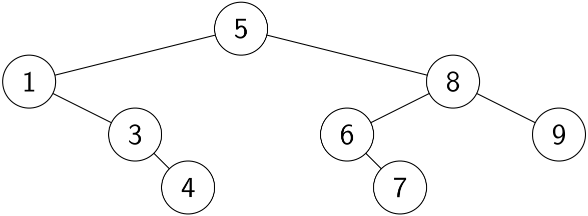


Binary Search Tree

- It is easy to generate a worst case testcase
- How to get rid of the $O(N)$ worst case?
- Use variants of BST
  - AVL tree
  - Red-black tree
  - AA tree
  - Treap
  - Size-balanced tree
  - Splay tree

Implementation

```
int root, value[10000], left[10000], right[10000];
```



	0	1	2	3	4	5	6	7		
value	1	6	5	8	3	9	7	4		
left			0	1						
right	4	6	3	5	7					

Implementation

or, if you know C++ and pointer very well,

```
struct BST {
  int value;
  BST *left, *right;
} *root, memory[10000], memory_i;
```

Avoid `malloc()` because it is slow

C++ Standard Template Library

- `set`, `map`, `multiset`, `multimap`
- Red-black tree internally
- worst case $O(\log N)$

```
#include <set>
...
std::set<int> s;
s.insert(456);
int x = s.count(456); //1
x.erase(456);
```

```
#include <map>
...
std::map<int, int> m;
m[456] = 123;
int x = m[456]; //123
m.erase(456);
```

Practice problems

- 01090 Diligent
- 01019 Addition II
- 30107 What is the Median?
- M0811 Alice's Bookshelf
- M0913 Is
- AP121 Dispatching

启发式合并

- Two literal meanings:
  - Insightful merging
  - Open-up merging (not this one)
- Similar to *Union by Rank* in disjoint-set union (Data Structures (III))
- Seems there is not a commonly used English term
- Problem: need to merge N items from small binary heaps into a large binary heap
- Idea: when merging two heaps, insert the smaller one into the larger one
- Time complexity: $O(N \log^2 N)$

启发式合并

- Consider an item X
- Initially X belongs to a heap of size 1
- Whenever the heap containing X is merged into a larger heap, the size of the heap is at least doubled
- So X can be inserted into another heap for at most $O(\log N)$ times
- Each of the N items can be inserted into another heap for at most $O(\log N)$ times
- Total number of heap insertions $= O(N \log N)$
- Time complexity $= O(N \log^2 N)$