Dynamic Programming (I)

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Reminder

- Assuming all variables are initialized to zero in all the codes appearing in this powerpoint

Perquisites – things learned before

- Recursion
- Divide and Conquer
- Time complexity analysis (using big O notation)

Fibonacci sequence - recall

- Define:
  - $f(1) = f(2) = 1$
  - $f(n) = f(n - 1) + f(n - 2)$ for $n > 2$
- How can we compute $f(n)$?

Fibonacci sequence - recursion

- We can use recursion and the definition of f(n).
- Code:

```
int f (int n) {
    if (n == 1 || n == 2) return 1;
    else return f(n - 1) + f(n - 2);
}
```

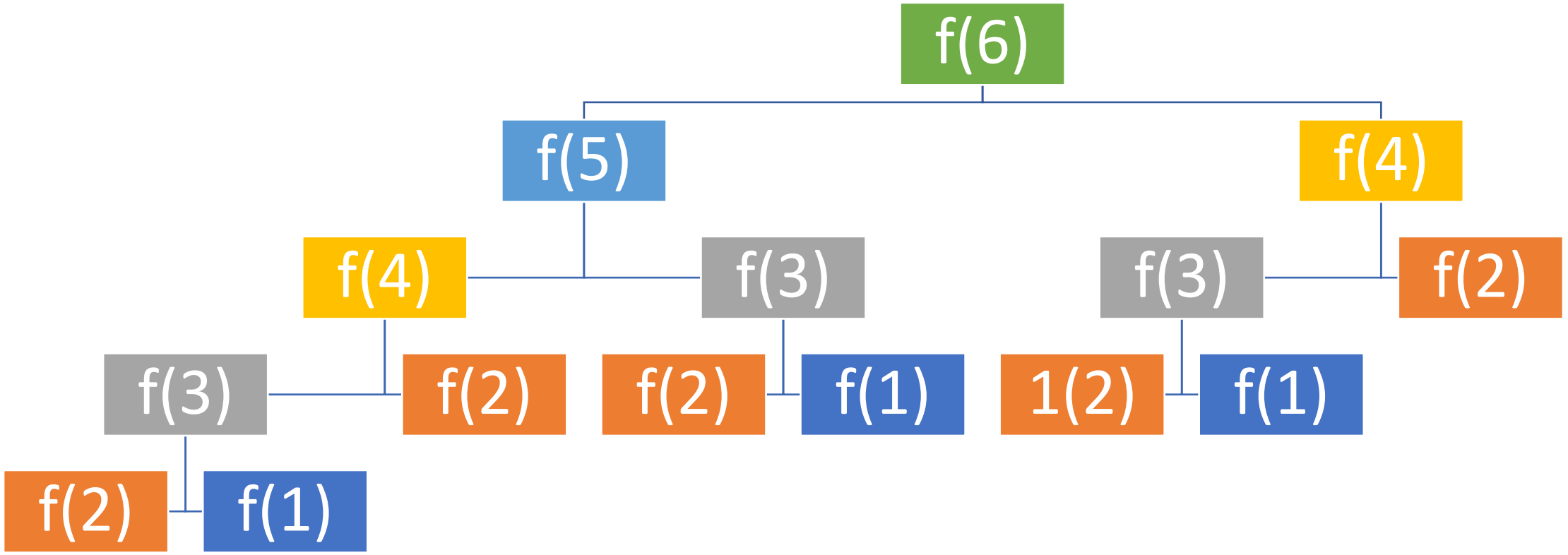
Fibonacci sequence - recursion

- Time complexity?
- $T(n) = T(n - 1) + T(n - 2) + 1 = O(f(n))$ (or simply $O(2^n)$)
- And we know f(n) grows pretty quickly, so this algorithm runs slow
- But why? Can we speed up the computation?

Fibonacci sequence - recursion

- Lets see how do we compute f(6)

Fibonacci sequence - recursion



Fibonacci sequence - recursion

- As we can see, most of the functions are evaluated more than once
- We say these computing power is wasted
- How can we improve the algorithm?

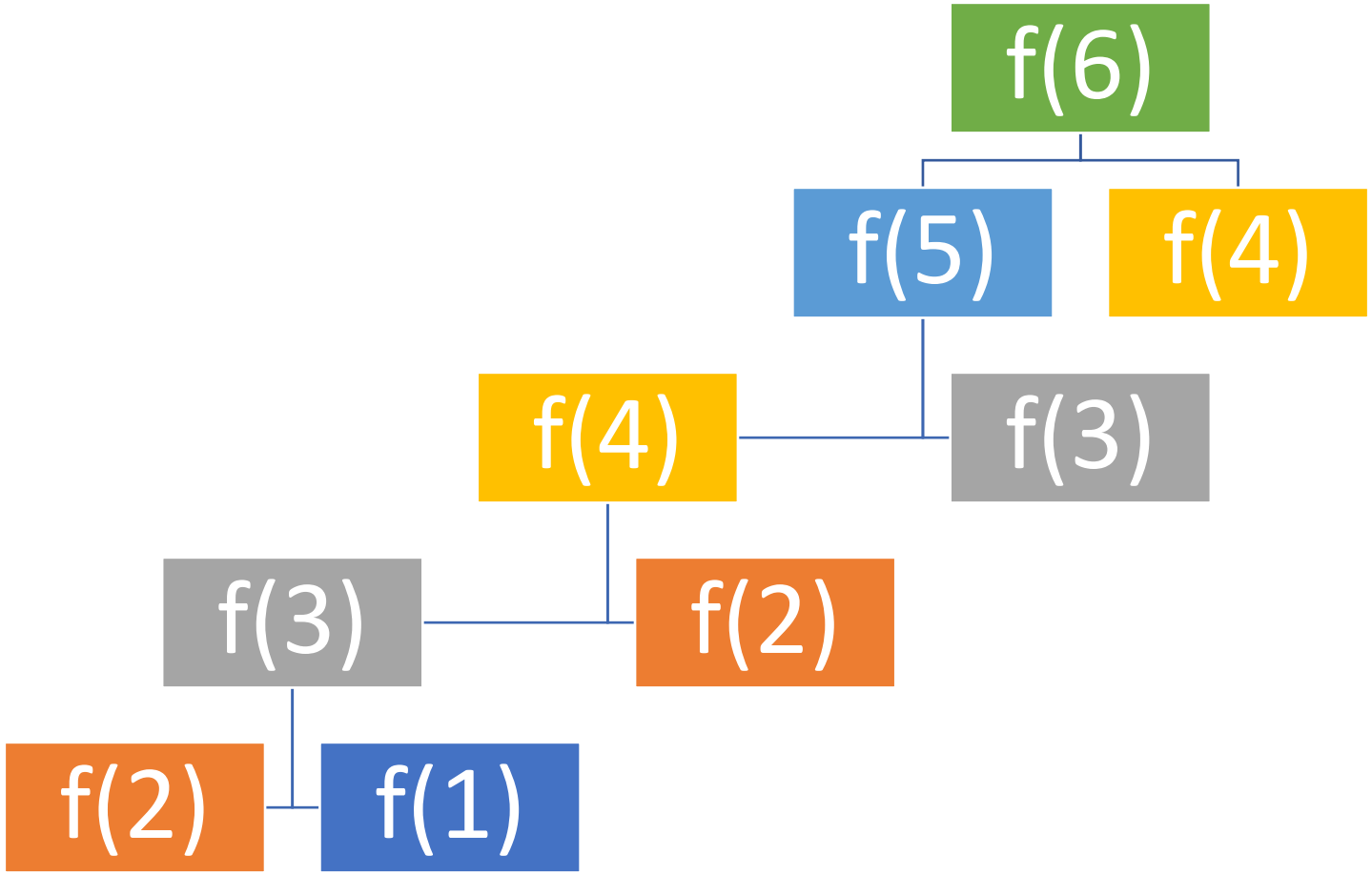
Fibonacci sequence - memorization

- One way to optimize is to store the evaluated value of the function so that we do not need to compute it again
- Code:

```
int f (int n) {
    if (dp[n]) return dp[n];
    if (n == 1 || n == 2) return 1;
    else return dp[n] = f(n - 1) + f(n - 2);
}
```

- Lets see how this algorithm looks like

Fibonacci sequence - memorization



Fibonacci sequence - memorization

- As we can see, we now reduced our computation a lot.
- What is the time complexity now?
- $T(n) = T(n - 1) + 1 = O(n)$
- Wow, now we can compute the Fibonacci sequence in very short time
- But remember, f(n) grows pretty fast, so if n is large, you may need to use long long or HPA for the calculations

Fibonacci sequence - memorization

- Here, we use a top down approach to solve the problem
- ie. We try to find the solution to our problem (f(n)) directly
  we find the solution of its sub-problem (f(n - 1) and f(n - 2)) whenever we need and record it down for future usage
- But why does it work?
- It’s because that the sub-problem (f(n - 1) and f(n - 2)) of our problem (f(n)) can be calculated in the same manner so that we can use a recursion to get the work done

Dynamic programming

- No we are learning a powerful technique, dynamic programming
- A problem can be solved using dynamic programming should fulfill:
  - It have optimal substructure
    - An optimal solution of a problem can be constructed efficiently form the optimal solutions of its sub-problems
  - It have overlapping sub-problems
    - Can be broken down into sub-problems which will be reused several times; Or
    - A recursive algorithm for the problem can solve the sub-problems
- Also we will define a ‘state’ for every sub-problem and we need to make sure that the solution to a certain sub-problem only depends on the its ‘state’

Fibonacci sequence – bottom up DP

- Let define dp[n] to be f(n)
- We then have the base cases dp[1] = dp[2] = 1
- And the recurrence relationship of the problems (aka. transition formula) is
  - dp[n] = dp[n - 1] + dp[n - 2] for n > 2
- So we can use a for loop to do this
- Code:

```
dp[1] = dp[2] = 1;
for (int i = 3; i <= n; i++) dp[n] = dp[n - 1] + dp[n – 2];
```

- When you apply the transition formula, make sure that the sub-problems is **already solved** when using the bottom up approach

Maximum subarray sum - problem

- Same as HKOI Judge 01010 Diamond Chain
- Let $a_1, a_2, a_3, \ldots, a_n$ be a sequence with length n
- Define subarray of the sequence to be the list $a_i, a_{i+1}, \ldots, a_{j-1}, a_j$ for some $i < j$
- The problem is to find the maximum sum of the subarray if we choose i, j in a optimal way
- If all a_i are negative, the answer is zero (ie. an empty subarray)

Maximum subarray sum – brute force

- We can use brute force to find the answer
- Code:

```
for (int i = 1; i <= n; i++) for (int j = i; j <= n; j++) {
                sum = 0;
                for (int k = i; k <= j; k++) sum += a[k];
                ans = max(ans, sum);
}
```

- Time complexity?
- $O(n^3)$

Maximum subarray sum – brute force

- We can use partial sum (aka. prefix sum) to optimize it
- Code:

```
for (int i = 1; i <= n; i++) sum[i] = sum[i - 1] + a[i];
for (int i = 1; i <= n; i++) for (int j = i; j <= n; j++)
    ans = max(ans, sum[j] – sum[i - 1]);
```

- Time complexity?
- $O(n^2)$
- Can it be faster?
- Yes, we can use DP, how?

Maximum subarray sum - DP

- When doing DP problems, we often define the state to be something like "The optimal solution from 1 to i" or "The optimal solution containing i"
- Here, we use the latter one
- We define dp[i] to be the maximum subarray sum where a_i is the last element of the subarray
- ie. dp[i] = $\max_{1 \le j \le i} \sum_{k=j}^{i} a_k$
- And the answer to the problem is $\max_{1 \le i \le n} dp[i]$
- Here, the optimal solution is either concatenate the next element to the previous one or let it be the only element (why?)
- So, our transition formula is
- dp[i] = max(dp[i - 1] + a[i], a[i])

Maximum subarray sum - DP

- Code:

```
for (int i = 1; i <= n; i++) dp[i] = max(dp[i - 1] + a[i], a[i]);
for (int i = 1; i <= n; i++) ans = max(ans, dp[i]);
```

- Time complexity?
- O(n)

Maximum subarray sum - DP

“I have learned that we can use divide and conquer to solve the same question with the same time complexity, why should I learn DP?”

- The code is shorter which bring a lot of benefits
  - Easier to debug
  - Quicker to code
- DP is guaranteed to give us the optimal solution if we define the state and transition formula correctly
- However, the divide and conquer solution is needed if we are finding the online maximum subarray sum in which case DP is not the fastest solution (HKOI Judge M0923, this requires some data structure)

Jumper - problem

- In a city, there is N buildings, counting from the left, the i^{th} building has height h_i
- There is a man who want to jump from building 1 to building N
- When he jump from building i to building j, the energy consumed is $|h_i - h_j|$
- Although he can jump very high, but he is so scared that he can only jump through at most K gaps between the buildings
- Also, he can only jump to the right
- What is the minimum energy consumption?

Jumper - DP

- Define dp[i] be the minimum energy consumption for the man to jump to building i
- The transition formula is
- $dp[i] = \min_{i-k \le j < i} (|h[i] - h[j]| + dp[j])$
- Since the man can only jump from a building between i – k and i, so we just need to find the minimum one

Jumper - DP

- Code:

```
for (int i = 2; i <= N; i++) {
    dp[i] = INF;
    for (int j = max(1, i- k); j < i; j++)
        dp[i] = min(dp[i], dp[j] + abs(h[i] – h[j]));
}
```

- Time complexity?
- O(NK)

Longest common subsequence - problem

- Given two string S and T, find the length of longest common subsequence
- Here, subsequence of a string S is any string that can be obtained by removing some(possibly none) characters of S

- EG.
- S = “ABCDEFG”
- T = “XVBDGH”
- Longest common subsequence = “BDG” (length 3)

Longest common subsequence – brute force

- It is not an easy job to think of a brute force solution for the problem
- Perhaps we should go ahead to think of the DP solution
- Remember what I have talked before
- When doing DP problems, we often define the state to be something like “The optimal solution from 1 to i” or “The optimal solution containing i”
- This time, we will choose the former one
- But since now we have 2 strings, so we need a 2D array

Longest common subsequence - DP

- Define dp[i][j] be the length of longest common subsequence of string S[0], S[1], …, S[i] and string T[0], T[1], …, T[j]
- Transition formula:
- dp[i][j] = max(dp[i - 1][j], dp[i][j -1]) if S[i] != T[j]
- dp[i][j] = max(dp[i - 1][j], dp[i][j – 1], dp[i - 1][j - 1] + 1) if S[i] == T[j]

- If S[i] != T[j], then the answer will be just the same as we either concatenate S[i] to S[0], S[1], … S[i - 1] or concatenate T[j] to T[0], T[1], …, T[j - 1] without changing another string
- But if S[i] == T[j], there may be one more option, that is to concatenate both strings at the same time, and increase the previous answer by 1

Longest common subsequence - DP

- Code:

```
for (int i = 1; S[i]; i++) for (int j = 1; T[j]; j++) {
        dp[i][j] = max(dp[i - 1][j], dp[i][j - 1]);
        if (S[i] == T[j]) dp[i][j] = max(dp[i][j], dp[i - 1][j - 1] + 1);
}
```

- Time complexity?
- O(|S||T|)

A short break

Knapsack problem - problem

- Given N items, each with a weight w_i and value v_i
- Our bag have a capacity of K which means for any set of items, we can put them in iff their total weight is at most K
- We want to maximize the total value of items that we put into our bag
- Noted that the order of putting items does not matter
- for each item, we either put it in or not put it in

Knapsack problem – brute force

- We can use brute force to find the answer
- Code:

```
for (int i = 1; i <= (1 << N); i++) {
        sum_w = 0;
        sum_v = 0;
        for (int j = 0; j < N; j++) {
                if (i & (1 << j)) {
                        sum_w += w[j];
                        sum_v += v[j];
                }
        }
        if (sum_w <= K) ans = max(ans, sum_v);
}
```

Knapsack problem – brute force

- Time Complexity?
- $O(2^N)$

- Seems very slow…
- How can we do faster?

Knapsack problem – wrong attempt

- Perhaps we may define the efficiency of each item e_i to be v_i/w_i
- And perhaps we may put the most efficient item into our bag, then the second and so on
- However, this is wrong because we cannot cut our item into pieces
- Can you think of a counter example?

Knapsack problem – DP

- Define dp[i][j] to be the largest total value of items using item 1, item 2, up to item i with total weight at most j.
- The answer is dp[N][K]
- The transition formula is
- dp[i][j] = max(dp[i - 1][j], dp[i - 1][j – w[i]] + v[i])

- This means we are either putting or not putting the item into our bag

Knapsack problem – DP

- Code:

```
for (int i = 1; i <= N; i++) for (int j = w[i]; j <= K; j++)
                dp[i][j] = max(dp[i][j], dp[i - 1][j – w[i]] + v[i]);
```

- Time complexity?
- O(NK)

Knapsack problem - Variations

- What if:
  - Additionally, the bag has a capacity of volume M and each item has volume u_i?
  - We can pick as many item i as we want?
  - We can pick at most p_i of the item i
  - We can pick item i only if we have picked item j for some i?

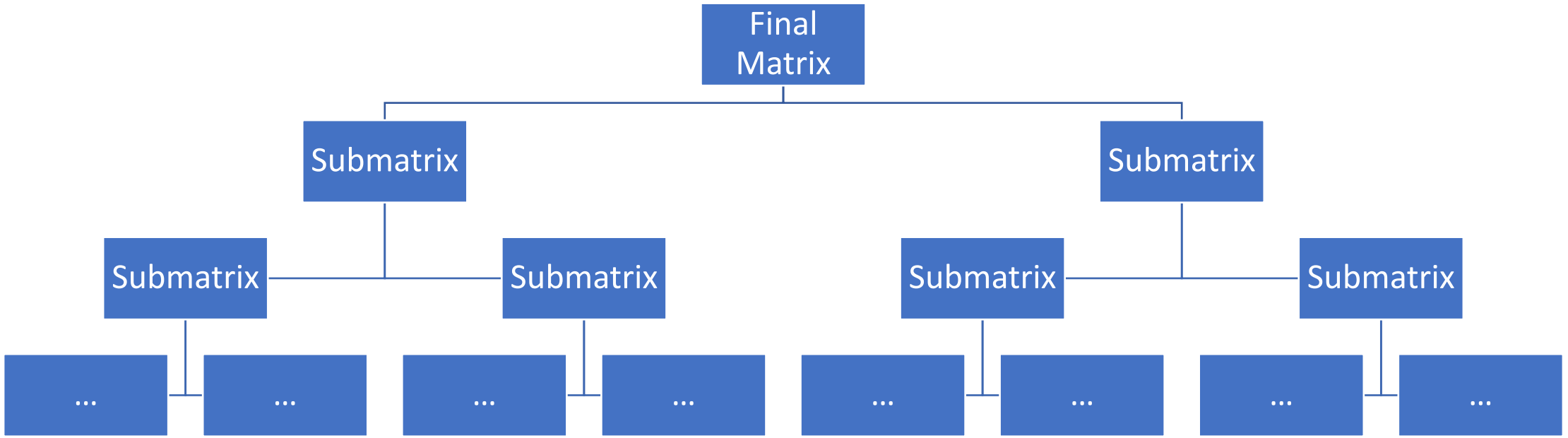
Matrix multiplication - problem

- HKOI Judge 01054
- A m×n matrix is like a 2D array of numbers with m rows and n columns
- Let A_1 and A_2 be matrices, A_1A_2 exist iff no. of columns in A_1 equals to no. of rows in A_2
- Let say, A_1 is an m×p matrix and A_2 is an p×n matrix, then A_1A_2, the multiple of two matrices, will be a m×n matrix, the cost of such operation is m×p×n
- Matrix multiplication is associative, ie. $(A_1A_2)A_3 = A_1(A_2A_3)$
- But matrix multiplication is not commutative, ie. A_1A_2 != A_2A_1 generally
- Given N + 1 numbers, namely p_1, p_2, … p_{n+1}, let A_i be a $p_i \times p_{i+1}$ matrix, find the minimum cost to do the matrix chain multiplication

Matrix multiplication - problem

- Eg.
- A_1 has dimension 5×10
- A_2 has dimension 10×15
- A_3 has dimension 15×5
- If we calculate $(A_1A_2)A_3$, the total cost is $5 \times 10 \times 15 + 5 \times 15 \times 5 = 1125$
- If we calculate $A_1(A_2A_3)$, the total cost is $10 \times 15 \times 5 + 5 \times 10 \times 5 = 1000$
- So the minimum cost is 1000

Matrix multiplication



Matrix multiplication - DP

- How can we define the states?

Matrix multiplication - DP

- How can we define the states?
- We can define dp[i][j] to be the minimum cost to multiply $A_iA_{i+1}…A_j$
- And the answer is dp[1][N]
- Then what is the transition formula?

Matrix multiplication - DP

- How can we define the states?
- We can define dp[i][j] to be the minimum cost to multiply $A_iA_{i+1}…A_j$
- And the answer is dp[1][N]
- Then what is the transition formula?
- $dp[i][j] = \min_{i<k\leq j}(dp[i][k-1] + dp[k][j] + p[i]\times p[k]\times p[j+1])$
- For every i, j, we are finding the optimal way to cut them into two submatrices
- We now have both the states and the transition formula but how can we actually calculate all dp[i][j]?

Matrix multiplication – Wrong DP

- Code:

```
for (int i = 1; i <= N; i++) for (int j = i + 1; j <= N; j++) {
        dp[i][j] = INF;
        for (int k = i + 1; k <= j; k++)
                dp[i][j] = min(dp[i][j], dp[i][k - 1] + dp[k][j] + p[i] * p[k] * p[j + 1])
}
```

- Why is this wrong?

Matrix multiplication – Wrong DP

- Code:

```
for (int i = 1; i <= N; i++) for (int j = i + 1; j <= N; j++) {
        dp[i][j] = INF;
        for (int k = i + 1; k <= j; k++)
                dp[i][j] = min(dp[i][j], dp[i][k - 1] + dp[k][j] + p[i] * p[k] * p[j + 1])
}
```

- Remember what I have said before:
- When you apply the transition formula, make sure that the sub-problems is **already solved** when using the bottom up approach

Matrix multiplication - DP

- Due to the nature of this question, it is a little bit hard to design the order of dp[i][j] that we need to calculate, it will be better if we use a top down approach

Matrix multiplication – top down DP

- Code:

```
long long solve(int i, int j) {
    if (i == j) return 0;
    if (v[i][j]) return dp[i][j];
    v[i][j] = true;
    dp[i][j] = INF;
    for (int k = i + 1; k <= j; k++)
        dp[i][j] = min(dp[i][j], solve(i, k - 1) + solve(k, j) + p[i] * p[k] * p[j + 1]);
    return dp[i][j];
}
```

Matrix multiplication – top down DP

- Time complexity?
- No. of states: $O(n^2)$
- State transition: $O(n)$
- Total: $O(n^3)$

Practice problems

Let’s think of the solutions

Palindrome subsequence - problem

- Given a string S with length N
- You are going to find the length of the longest palindrome subsequence
- A subsequence of a string S is any string that can be obtained by removing some(possibly none) characters in S
- A palindrome is a string that looks the same after it is reversed

- How can we define the state and the transition formula?
- What is the time complexity of the algorithm?

Palindrome subsequence – solution 1

- Let T be a string obtained by reversing S.
- The answer is simply the longest common subsequence of S and T

- Why is this true?

- Think of another solusion

Palindrome subsequence – solution 2

- We will use the top down approach
- States:
  - Define dp[i][j] to be the length of maximum length of palindrome subsequence for the substring S[i], S[i+1], …, S[j]
- Transition formulars:
  - dp[i][j] = dp[i+1][j-1] + 2 if S[i] == S[j] (this means we cut the first and the last character)
  - dp[i][j] = max(dp[i+1][j], dp[i][j - 1]) if S[i] != S[j] (this means we cut either the first or the last character)
- Base cases:
  - dp[i][j] = 0 if i > j
  - dp[i][j] = 1 if i == j
- Time complexity: $O(n^2)$

Balanced parentheses - problem

- Given the length of the string is 2N
- Find the no. of different strings such that it contain exactly N open brackets ‘(’ and N close brackets ‘)’ and it is balance
- We say the parentheses are balanced if the no. of open brackets is not less than the no. of close brackets in any prefix.
- Since the number is large, you are only required to output the answer mod $10^9 + 7$

- How can we define the state and the transition formula?
- What is the time complexity of the algorithm?

Balanced parentheses - solution 1

- State:
  - Define dp[i][j] to be the no. of strings (mod $10^9 + 7$) with length i, containing exactly j open brackets (ie. containing exactly i – j close brackets) and for any of its prefix, no. of open brackets is no less than no. of close brackets
- Transition formula:
  - dp[i][j] = dp[i - 1][j - 1] (we can put an open bracket)
  - dp[i][j] += dp[i – 1][j] if j > (i - 1) - j (we can also put a close bracket)
  - Remember to mod $10^9 + 7$
- Base case:
  - dp[1][1] = 1 (or dp[0][0] = 1)
- Time complexity: $O(n^2)$

Balanced parentheses - solution 2

- State:
  - Define dp[i] to be the no. of strings (mod $10^9 + 7$) with length 2i such that the string of parentheses is balanced.
- Transition formula:
  - $dp[i] = \sum_{j=1}^{i} dp[j-1]dp[i-j]$
  - This is to cut the string into a prefix and suffix (first 2j and last 2(i-j)) such that the prefix is balanced after removing the pair of outermost parentheses and the suffix is balanced
  - Remember to mod $10^9 + 7$
- Base case:
  - dp[1] = 1 (or dp[0] = 1)
- Time complexity: $O(n^2)$
- The sequence dp[1], … dp[n] is also known as catalen number

More problems:

- Maximum subarray sum
  - 01010 Diamond Chain
  - 01016 Diamond Ring
  - M0822 Diamond Chain II
- Knapsack problem
  - 05011 Coin
  - T043 Need for speed
- Palindrome
  - I0011 Palindrome
  - CF 607B Zuma

More problems:

- Parentheses
  - CF 629C Famil Door and Brackets
- Combinatorics
  - CF 553A Kyoya and Colored Balls
- Probabilities
  - CF 540D Bad Luck Island
- Extra
  - M1724 Guess the Number
    - Easier to solve some kind of ‘reverse’ problem