

# Dynamic Programming (III)

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# Today's highlights

- Ugly state DP
  - DP Optimization
- 
- Note: Some materials from DP (III) in 2015 are not covered in this lesson. You are encouraged to study them yourself.

# Ugly state DP

- We have seen “nicer” DP problems
  - For example, count number of paths from (1,1) to (N,M)  
if (i,j) is blocked  
     $dp[i][j] = 0;$   
else  
     $dp[i][j] = dp[i-1][j] + dp[i][j-1]$
- But the state representation is not always nice!
  - For example, bitmask DP

# NOI 2001 炮兵陣地

- Given a  $N \times M$  grid ( $N \leq 100$ ,  $M \leq 10$ )
- Some cells are hills ('H'), some plains ('P')
- You can place artillery (炮兵) only in 'P' cells
- Artillery squads cannot attack each other
- Their attack range:

..X..

..X..

XXPXX

..X..

..X..

# NOI 2001 炮兵陣地

- Your task is to find the maximal number of artillery squads that may be placed
- Notice that  $M$  is small
- How to DP?

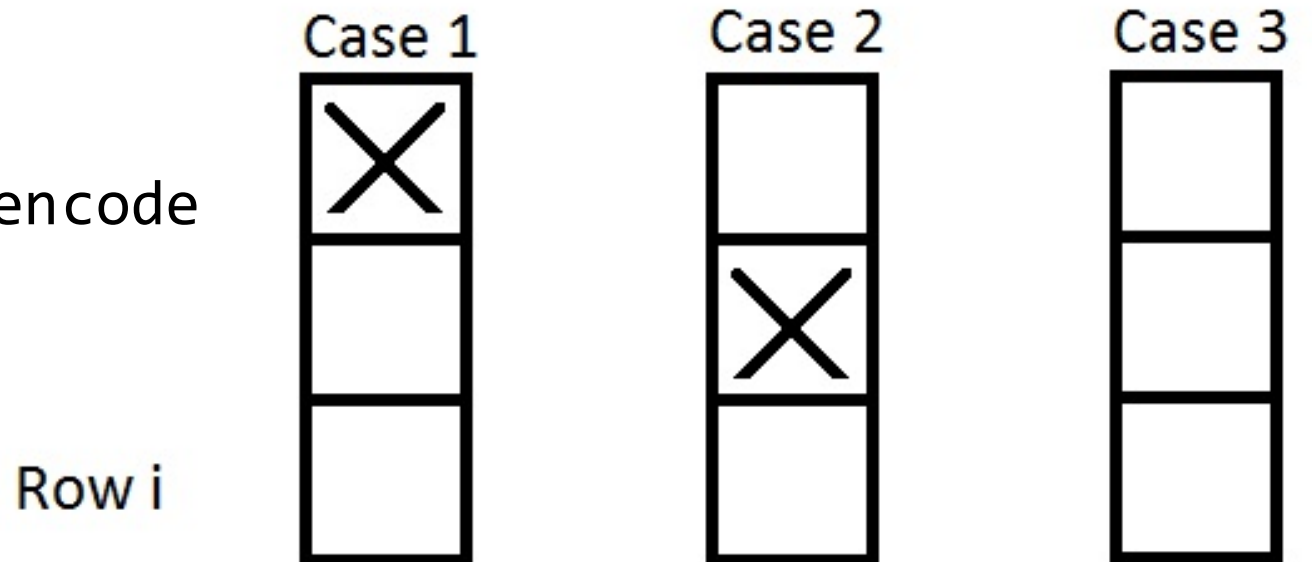
# Analysis

- Proceed row by row
- Assume the squads on row 1 ... (i-1) are fixed
- One can place a squad in cell (i, j) iff:
  - [Previous rows] Cells (i-1, j) and (i-2, j) are empty
  - [Current row] Cells (i, j-2), (i, j-1), (i, j+1) and (i, j+2) are empty
  - [Current cell] Cell (i, j) is 'P'

# Representing the states

- [Previous rows]
- We see that squad placement on rows  $1 \dots (i-3)$  does not affect row  $i$
- Only need to know the “state” of the previous two rows

- Three possibilities
  - Use base-3 integers to encode
  - $0 = \text{Case 1}$
  - $1 = \text{Case 2}$
  - $2 = \text{Case 3}$



# State transition

- [Current row]
- When we do state transition, we select a subset of  $\{1, 2, \dots, M\}$  to put squads on
- To prevent attacking each other, make sure adjacent values differ by at least 3
- $\{1, 4, 8\}$  is ok,  $\{\}$  is ok, while  $\{1, 5, 7\}$  is not



# Algorithm

```
for i from 1 to N
  for j from 0 to 3^M - 1
    for all valid row patterns //e.g. {1, 4, 8}
      if (all corresponding positions are 'P') and
        (none attacked by squads on previous rows)
        j' = new state
        c = size of row pattern //{1, 4, 8} → c = 3
        dp[i][j'] = max(dp[i][j'], dp[i-1][j] + c)
```

# Summary

- Time complexity:  $O(3^M * N * (\# \text{ row patterns}))$
- Implementation details may be complicated
- If you want to submit:
  - <http://wcipeg.com/problem/noi01p2>
  - <http://poj.org/problem?id=1185>

(both require registration, of course)

# Further optimization

- Skip the state if  $dp[i-1][state] = \text{NINF}$
- Not many states are reachable due to the restriction on same row placement
- e.g.  $221210_3$  is not reachable

# With Optimization

```
for i from 1 to N
  for j from 0 to 3^M - 1
    if dp[i-1][j] = NINF
      continue
    for all valid row patterns //e.g. {1, 4, 8}
      if (all corresponding positions are 'P') and
        (none attacked by squads on previous rows)
        j' := new state
        c := size of row pattern //{1, 4, 8} → c = 3
        dp[i][j'] := max(dp[i][j'], dp[i-1][j] + c)
```

# Practice Problems (Ugly State DP)

- CF342D Xenia and Dominoes
- HK0J T112 Tetrisudoku

# Next: DP Optimization

- Monotone Queue
- Convex Hull Trick
- Divide and Conquer

# Why Optimize?

- Suppose you have a correct DP formula (state definition, state transition)
- Sometimes, the DP may not run in time!
- Four main ways to solve:
  - Explore non-DP solutions
  - Write auxiliary DPs (DP2[][], DP3[][], ...) to speed up
  - Come up with alternative DP formulas
  - Optimize DP transition → What we will explore today

# DP Optimization

- Monotone Queue



# Optimize using Monotone Queue

- The basic form of DP formula:

$$dp[i] = \max_{L(i) \leq j < i} (dp[j]) + f[i]$$

- $L(i)$  is increasing
  - Side note: If  $L(i)$  is not increasing, use segment tree to store, retrieve, and update dp values
- May replace  $dp[j]$  by any function depending on  $j$ 
  - e.g.  $g(j) = dp[j] * 2 - j$

# Optimize using Monotone Queue

- Naïve implementation:  $O(N^2)$

```
for i from 1 to N
```

```
    dp[i] = -INF
```

```
    for j from L(i) to i - 1
```

```
        dp[i] = max(dp[i], f(i) + g(j))
```

- We can do better!

# Bowling for Numbers ++

- CCC 2007 Stage 2 Problem
- You have  $N$  ( $N \leq 10000$ ) bowling pins and  $K$  ( $K \leq 500$ ) bowling balls, each ball has width  $w$  ( $w \leq 100$ )
- Each pin has a score  $s[i]$  from  $-10000$  to  $10000$
- You are allowed to miss
- Maximum achievable score = ?

# Bowling for Numbers ++

Sample (N = 9, K = 4, w = 3)

- 2 8 -5 3 5 8 4 8 -6
  - X X -5 3 5 8 4 8 -6 (ball 1, score = 10)
  - \_ \_ -5 X X X 4 8 -6 (ball 2, score = 26)
  - \_ \_ -5 \_ \_ X X X -6 (ball 3, score = 38)
  - \_ \_ -5 \_ \_ \_ \_ \_ -6 (ball 4, score = 38)
- Answer = 38

# Bowling for Numbers ++

- What if all pins have non-negative values?
  - Then, it is always better to hit more pins than to miss
  - $dp[i][j]$ : max. value if we use  $i$  balls to hit pins  $1\dots j$
  - $dp[i][j] = \max(dp[i][j-1], dp[i-1][j-W] + \text{sum}(s[j-W+1]\dots s[j]))$  //roughly
- $O(NK)$
- CCC 2007 Stage 1 Senior Q5

# Bowling for Numbers ++

- Unfortunately, such is not the case in the current task
- We may sometimes want to “avoid” negative values

# DP State definition

- Let  $ps[i] = s[1] + s[2] + \dots + s[i]$ 
  - Partial sum
- Let  $dp[i][j]$  be the maximum achievable score with  $i$  hits and the rightmost hit pin being  $j$
- $dp[0][0] = 0$
- $dp[0][i] = -INF$  for  $i > 0$ 
  - Otherwise, error --- for example, a negative pin may be “skipped” without cost

# Two cases

- Let  $dp[i][j]$  be the maximum achievable score with  $i$  hits and the rightmost hit pin being  $j$
- We assume the hits are from left to right
- Two cases:
  - Case 1. The  $i^{\text{th}}$  hit does not overlap with the  $(i-1)^{\text{th}}$  hit
  - Case 2. The  $i^{\text{th}}$  hit overlaps with the  $(i-1)^{\text{th}}$  hit



# Transition formula

- To calculate  $dp[i][j]$ ,
- Let  $M1 = \max_{0 \leq k < j-w} (dp[i-1][k] + ps[j] - ps[j-w])$ 
  - Case 1
  - Use  $dp2[i-1][k] := \max(dp[i-1][1], \dots, dp[i-1][k])$
- Let  $M2 = \max_{j-w \leq k < j} (dp[i-1][k] + ps[j] - ps[k])$ 
  - Case 2
  - Use monotone queue to optimize!
- Then  $dp[i][j] = \max(M1, M2)$

# Monotone Queue: Step by step

- Here is how the monotone queue (actually a deque) works.
- Recall the criteria:
  - $dp[i] = \max_{L(i) \leq j < i} (f(i) + g(j))$
  - $L(i)$  is increasing
- We maintain a queue  $Q[]$  of indices such that
  - $Q[j] < Q[j+1]$
  - $g(Q[j]) \geq g(Q[j+1])$for all  $j$ .

# Monotone Queue: Step by step

- Step 1: Pop elements in the front that are “out of bounds”

```
while (Q not empty) and (Q[l] < L(i))  
    l++;
```

(Here, we use an array and two pointers,  $l$  and  $r$ , to represent the deque.)

# Monotone Queue: Step by step

- Step 2: Update answer using  $Q[1]$

```
if (Q not empty)
    dp[i] = f(i) + g(Q[1]);
```

# Monotone Queue: Step by step

- Step 3: Pop elements at the back that have small values

```
while (Q not empty) and (g(Q[r]) < g(i))  
    r--;
```

# Monotone Queue: Step by step

- Step 4: Insert  $i$  at the back

$Q[++r] = i;$

# Monotone Queue: Step by step

1. while (Q not empty) and ( $Q[l] < L(i)$ )  
     $l++$ ;
2. if (Q not empty)  
     $dp[i] = f(i) + g(Q[l])$ ;
3. while (Q not empty) and ( $g(Q[r]) < g(i)$ )  
     $r--$ ;
4.  $Q[++r] = i$ ;

# Summary

- Original time complexity:  $O(N^2K)$  /  $O(NKw)$
- Optimized time complexity:  $O(NK)$
- Monotone queue optimization reduces DP time complexity by an order of N
- We can use rolling array to reduce memory complexity to  $O(N)$



# Final step before AC

- $N = 2, K = 1, w = 2$
- -1 1
- Answer = 1, but cannot get from our dp...
- Solution: add  $(w - 1)$  copies of 0s at the end

```
break;
```

# DP Optimization

- Monotone Queue
- Convex Hull Trick

# Convex Hull Trick (CHT)

- The earliest CHT task that I know is “Batch Scheduling” in IOI 2002
  - 11 contestants got full scores :o
- Other occurrences in big competitions:
  - APIO 2010 Commando
  - APIO 2014 Split the Sequence
- Something a good IOI contestant should know :D

# Convex Hull Trick (CHT)

- The basic form of DP formula:

$$dp[i] = \max_{j < i} (dp[j] + f[i] * g[j])$$

- Intuitively looks like  $y = mx + c$ , a line on  $\mathbb{R}^2$
- We may apply CHT if  $g$  is monotone
- If  $f$  is also monotone, then it will be even easier

# CF189C Kalila and Dimna in the Logging Industry

- Problem reduces to:
- Given  $N$ ,  $a[i]$ ,  $b[i]$ , find indices  $p_1, \dots, p_k$  such that  $p_1 = 1$ ,  $p_k = N$ ,  $p_i < p_{i+1}$  for all  $i$ , and  $\text{sum}(a[p_{i+1}] * b[p_i])$  is minimal. Output that minimal sum.
- Important constraints:
  - $a[]$  is (strictly) increasing
  - $b[]$  is (strictly) decreasing

# CF189C Kalila and Dimna in the Logging Industry

- e.g.  $N = 6$
- $a[] = \{1, 2, 3, 10, 20, 30\}$
- $b[] = \{6, 5, 4, 3, 2, 0\}$
- Choose indices 1, 2, 4, 6  
→  $\text{sum} = a[2]*b[1] + a[4]*b[2] + a[6]*b[4] = 152$
- Choose indices 1, 3, 6  
→  $\text{sum} = a[3]*b[1] + a[6]*b[3] = 138,$   
which is better (and, in fact, the best)

# CF189C Kalila and Dimna in the Logging Industry

- Let  $dp[i]$  = the best result obtainable by picking indices ending with  $i$
- $dp[1] = 0$
- $dp[i] = \min_{j < i} (dp[j] + a[i] * b[j])$ 
  - Not max this time
  
- Naïve DP transition:  $O(N^2)$
- Target:  $O(N)$  !!



# Key idea of CHT

- Let us pick two indices  $j$  and  $k$  ( $1 \leq j < k < i$ )
- When will we choose  $j$  instead of  $k$ ? Or vice versa?

# Key idea of CHT

- When will we choose  $j$  instead of  $k$ ? Or vice versa?

$k$  better than  $j$

$$\Leftrightarrow dp[j] + a[i] * b[j] > dp[k] + a[i] * b[k]$$

$$\Leftrightarrow (dp[j] - dp[k]) / (b[j] - b[k]) > -a[i]$$

Does it look like  $(y_2 - y_1) / (x_2 - x_1)$  ?

$$\text{Let } m(j, k) = (dp[j] - dp[k]) / (b[j] - b[k])$$

# Key idea of CHT

k better than j

$$\Leftrightarrow (dp[j] - dp[k]) / (b[j] - b[k]) > -a[i]$$

Does it look like  $(y_2 - y_1) / (x_2 - x_1)$  ?

Let  $m(j, k) = (dp[j] - dp[k]) / (b[j] - b[k])$

(We assume  $j < k$  every time we write  $m(j, k)$ .)

# Key idea of CHT

$k$  better than  $j \iff m(j, k) > -a[i]$

- **Property 1:** If  $m(j, k) < m(k, l)$ , then there is no need to consider  $k$ .
- Reason: there are no cases where  $k$  must be chosen
- If  $m(j, k) > -a[i]$ , then surely  $m(k, l) > -a[i]$ .  
Then  $l$  is even better than  $k$
- If  $m(j, k) \leq -a[i]$ , then  $j$  is no worse than  $k$

# Key idea of CHT

$k$  better than  $j \iff m(j, k) > -a[i]$

- **Property 2:** If  $m(j, k) > -a[i]$ , there is no need to consider  $j$  in subsequent steps (steps  $i+1, \dots, N$ ).
- Reason:  $a$  is (strictly) increasing, so  
 $(-a[i]) > (-a[i'])$  for any  $i' > i$
- Then  $m(j, k) > -a[i] > -a[i']$

# Key idea of CHT

- **Property 1:** If  $j < k < l$  and  $m(j, k) < m(k, l)$ , then there is no need to consider  $k$ .
- **Property 2:** If  $m(j, k) > -a[i]$ , there is no need to consider  $j$  in subsequent steps (steps  $i+1, \dots, N$ ).
- What do these two properties give us?

# Key idea of CHT

- **Property 1:** If  $j < k < l$  and  $m(j, k) < m(k, l)$ , then there is no need to consider  $k$ .
- **Property 2:** If  $m(j, k) > -a[i]$ , there is no need to consider  $j$  in subsequent steps (steps  $i+1, \dots, N$ ).
- Property 1 means that we only need to maintain a deque  $q[L \dots R]$ , such that  $m(q[i], q[i+1]) \geq m(q[i+1], q[i+2])$

# Key idea of CHT

- **Property 1:** If  $j < k < l$  and  $m(j, k) < m(k, l)$ , then there is no need to consider  $k$ .
- **Property 2:** If  $m(j, k) > -a[i]$ , there is no need to consider  $j$  in subsequent steps (steps  $i+1, \dots, N$ ).
- Property 2 means that we can pop  $q[L]$  (front) from the deque, until  $m(q[L], q[L+1]) \leq -a[i]$ , to get the optimal answer



# CHT: Step by step

- Step 1: Pop elements in the front that we will never use again [Property 2]

```
while (q's size >= 2) and (m(q[L], q[L+1]) > -a[i])  
    L++;
```

# CHT: Step by step

- Step 2: Update answer using  $q[L]$

```
if (Q not empty)
```

```
     $dp[i] = dp[q[L]] + a[i] * b[q[L]];$ 
```

# CHT: Step by step

- Step 3: Pop elements at the back that will never be considered [Property 1]

```
while (q's size >= 2) and (m(q[R-2], q[R-1]) < m(q[R-1], i))  
    R--;
```

# CHT: Step by step

- Step 4: Insert  $i$  at the back

```
q[R++] = i;
```

# CHT: Step by step

1. while (q's size  $\geq 2$ ) and ( $m(q[L], q[L+1]) > -a[i]$ )  
    L++;
2. if (q not empty)  
     $dp[i] = dp[q[L]] + a[i] * b[q[L]]$ ;
3. while (q's size  $\geq 2$ ) and ( $m(q[R-2], q[R-1]) < m(q[R-1], i)$ )  
    R--;
4.  $q[R++] = i$ ;

# Summary

- Notice that CHT (at least in this problem) is simply a variant of monotone queue optimization
- The monotonicity does not lie in the values itself, but in the “slope function”
- Time complexity:  $O(N)$

# CHT: Further complications

- $dp[i] = \max_{j < i} (dp[j] + f[i] * g[j])$
- $f/g = i/i, i/d, d/i, d/d, n/i, n/d$ 
  - $i$ : increasing,  $d$ : decreasing,  $n$ : neither
  - $i/i$  and  $d/d$  are not interesting
  - For  $n/i$  and  $n/d$ , property 2 does not hold; **need b-search**
- Transition formula: max, min
- You have to write down the condition for “ $k$  better than  $j$ ” and do the algebra correctly

# CHT: Further complications

- When  $g$  is not strictly monotone (i.e. may have same values), direct computation of slope formula will give division by  $0$
- Also, using double for slope calculation may sometimes result in precision error



# DP Optimization

- Monotone Queue
- Convex Hull Trick
- Divide and Conquer

# Divide and Conquer (D&C) Optimization

- The basic form of DP formula:

$$dp[i][j] = \max_{k < j} (dp[i-1][k] + f(k, j))$$

- Let  $C[i][j]$  be the smallest index  $k'$  such that  $dp[i][j] = dp[i-1][k'] + f(k', j)$
- In other words, the transition from  $(i-1, k')$  to  $(i, j)$  is optimal among all choices of  $k$
- We can apply D&C Optimization if  $C[i][j] \leq C[i][j+1]$  for all  $j$

# D&C Optimization

- $C[i][j] \leq C[i][j+1]$  for all  $j$
- Another form of *monotonicity*!

# CF321E Ciel and Gondolas

- Given  $N$ ,  $G$ , and an  $N \times N$  symmetric matrix  $s[][]$  containing values from 0 to 9
- $s[i][i] = 0$  for all  $i$
- Divide  $[1, N]$  into  $G$  disjoint groups  $[1, a_1]$ ,  $[a_1 + 1, a_2]$ ,  $\dots$ ,  $[a_{G-1} + 1, a_G]$  ( $a_G = N$ )
- For each group  $[L, R]$ , calculate  $\text{sum}(s[i][j] \mid i \text{ and } j \text{ are between } L \text{ and } R)$
- Add them up to get the total cost of this partition
- Find the minimal cost

# CF321E Ciel and Gondolas

- 5 2 (N = 5, G = 2)
- 0 0 1 1 1
- 0 0 1 1 1
- 1 1 0 0 0
- 1 1 0 0 0
- 1 1 0 0 0
  
- Answer = 0 ( $a_1 = 2, a_2 = 5$ )

# CF321E Ciel and Gondolas

- Let  $dp[i][j]$  = minimal cost of partitioning  $[1, j]$  into  $i$  groups
- Let  $f(L, R) = \text{sum}(s[i][j] \mid i \text{ and } j \text{ are between } L \text{ and } R)$
- $dp[i][j] = \min_{k < j} (dp[i-1][k] + f(k+1, j))$
- Answer =  $dp[G][N]$

# CF321E Ciel and Gondolas

- $dp[i][j] = \min_{k < j} (dp[i-1][k] + f(k+1, j))$
- $f(k+1, j)$  can be calculated in  $O(1)$  by 2D partial sum
- Naïve solution has time complexity  $O(GN^2)$
- Target:  $O(GN \log N)$

# CF321E Ciel and Gondolas

- Recall the condition for applying D&C Optimization:
- Let  $C[i][j]$  be the smallest index  $k'$  such that  $dp[i][j] = dp[i-1][k'] + f(k', j)$ .
- **Check that  $C[i][j] \leq C[i][j+1]$  for all  $j$**



# D&C Optimization: Step by step

- The key idea is to write a recursive function to perform the DP transition
- `void solve(int i, int L, int R, int optL, int optR);`
- Let  $M = (L+R)/2$
- Find `dp[i][M]` and `C[i][M]`. Then call `solve()` for the left and the right parts.

# D&C Optimization: Step by step

- Step 1: Base case

```
if (L > R) return;
```

# D&C Optimization: Step by step

- Step 2: Find  $dp[i][M]$  and  $C[i][M]$

```
int opt = optL;          //opt represents C[i][M]
for(p = optL + 1; p <= optR; p++)
    if(dp[i-1][p] + f(p+1, M) < dp[i-1][opt] + f(opt+1, M))
        opt = p;
```

# D&C Optimization: Step by step

- Step 3: Update  $dp[i][M]$

$$dp[i][M] = dp[i-1][opt] + f(opt+1, M);$$

# D&C Optimization: Step by step

- Step 4: Recursively solve the left and right parts

```
solve(i, L, M - 1, optL, opt);  
solve(i, M + 1, R, opt, optR);
```

# D&C Optimization: Step by step

```
void solve(int i, int L, int R, int optL, int optR){  
1. if (L > R) return;  
2. int opt = optL;          //opt represents C[i][M]  
   for(p = optL + 1; p <= optR; p++)  
       if(dp[i-1][p] + f(p+1, M) < dp[i-1][opt] + f(opt+1, M))  
           opt = p;  
3. dp[i][M] = dp[i-1][opt] + f(opt+1, M);  
4. solve(i, L, M - 1, optL, opt);  
   solve(i, M + 1, R, opt, optR);  
}
```

# Back to the task...

- Set  $dp[0][0] = 0$  and  $dp[0][i] = \text{INF}$  for  $i > 0$
- Call  $\text{solve}(i, 1, N, 1, N)$  for  $i = 1, \dots, G$
- It can be shown that each  $\text{solve}()$  runs in time complexity  $O(N \log N)$
- Overall time complexity:  $O(GN \log N)$

# Minor details

- Use rolling array for DP calculation
- Huge input ( $4000^2$  numbers), need fast I/O methods to get AC



Any questions?

# Practice Problems (DP Opt)

- CF311B Cats Transport
- CF660F Bear and Bowling 4
- Hackerrank Guardians of the Lunatics
- APIO 2010 Commando
- APIO 2014 Splitting the Sequence
- ...and more in the CF blog mentioned in reference

# Other interesting DP Problems

- CF 590D Top Secret Task
- CF 489E Hiking
- HKOJ M1331 Resources Conflict
- HKOJ M1724 Guess the Number (:P)

# Reference

- Tasks from HKOJ, Codeforces, CCC, NOI
- <http://codeforces.com/blog/entry/8219>
  - A summary of different types of DP Optimization

# Appendix: Model solutions

- Bowling for Numbers ++
  - <http://ideone.com/tY1Kk3>
- Kalila and Dimna in the Logging Industry
  - <http://ideone.com/y47MiU>
- Ciel and Gondolas
  - <http://ideone.com/7Mpq2o>