# DP(II)

Anson Ho (ppt by Theo)

# Recall

Topics in DP(I)

what is the definition of “State”, “Base case” and “Transition formula”?

# Topics

- Memory optimization: Rolling array
- Bitwise DP
- Tree DP

# Rolling array

Recall the problem of fibonacci number:

f(0)=1
f(1)=1
f(a)=f(a-2)+f(a-1) if a>=2

Memory complexity: O(N)

# Rolling array

Memory complexity: O(N)->O(1)

useful ?

# Rolling array

Recall the path counting problem:

Given a N*M grid, with some grid blocked, find out the number of ways to move from (1,1) to (N,M) with right and down movement only.

# Rolling array

f[i][j]: way to go from (1,1) to (i, j).

f[1][1] = 1

f[i][j] = f[i - 1][j] + f[i][j + 1] if (a[i][j] is passable)

0 otherwise

Memory complexitiy? O(NM)

# Rolling array

if we have F[1..N][i], then we can have F[1..N][i+1] without referring to F[1..N][i-1]!

--> holds two column of “F” values is enough

O(NM) --> O(M)

# Rolling array

Even better --> if (M > N) swap N and M
O(M) --> O(min(N,M))

# Rolling array

Implementation 1
copy/swap the array everytimes

f[0]=f[1]
init(f[1])

# Rolling array

Implementation 2
exchange the role

When i is even, f[0]=transition(f[1])
When i is odd, f[1]=transition(f[0])

# Rolling array

But when do you need memory optimization?

# Rolling array

Case 1: your program is wasting memory

e.g.1 use counting sort when xi<=1e8

e.g.2 use array to store prime factorization
6->{0,0,1,1,0,0,…}

# Rolling array

Case 2: the problem has special condition

memory limit: 32MB
time limit: 4s

int x[3000][3000] –> MLE

# Rolling array

Case 2: the problem has special condition

But usually,
memory limit: 256MB
time limit: 1s

long long x[?] -> MLE

# Bitwise DP

Consider following problem:
Given a N*M grid, In how many ways can we place a 1*2 domino (in horizontal or vertial) so that the grid is completely filled and no domino overlap?

N<=1000, M<=10

# Bitwise DP

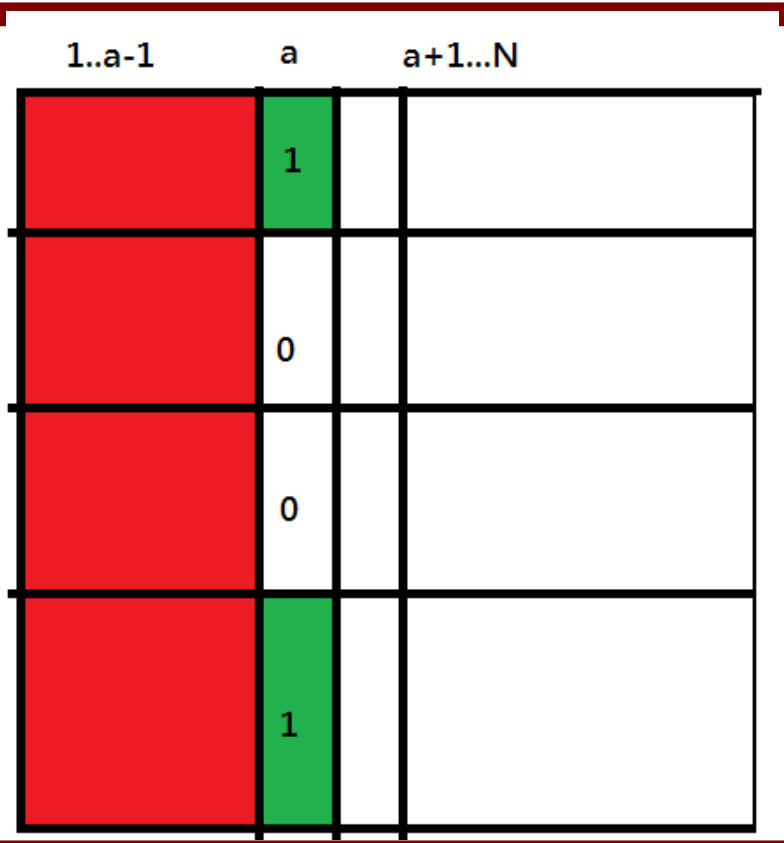
Notice that the placement of a domino on column M is determined by the domino config on column M-1 and M only.

--> dp[a][b]: number of ways to fill column 1..(a) such that all the columns 1..(a-1) is filled, and the shape of column a is b.

# Bitwise DP

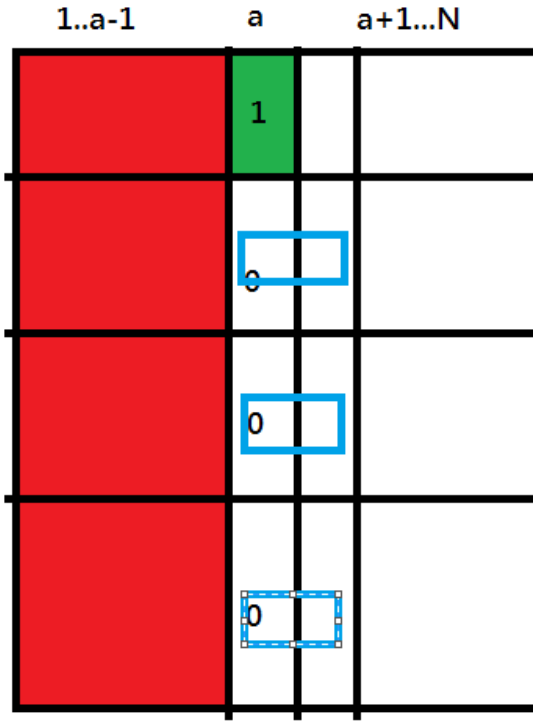
how to “define” a shape?

-->as we concerns about the filled “state” of the (a-1) column only, we can represent the shape as a binary number!



# Bitwise DP

Transition?
Given a shape, one can either put all the “0s” with horizontal dominos
-->dp[i][j] contributes to dp[i + 1][2^M-j]

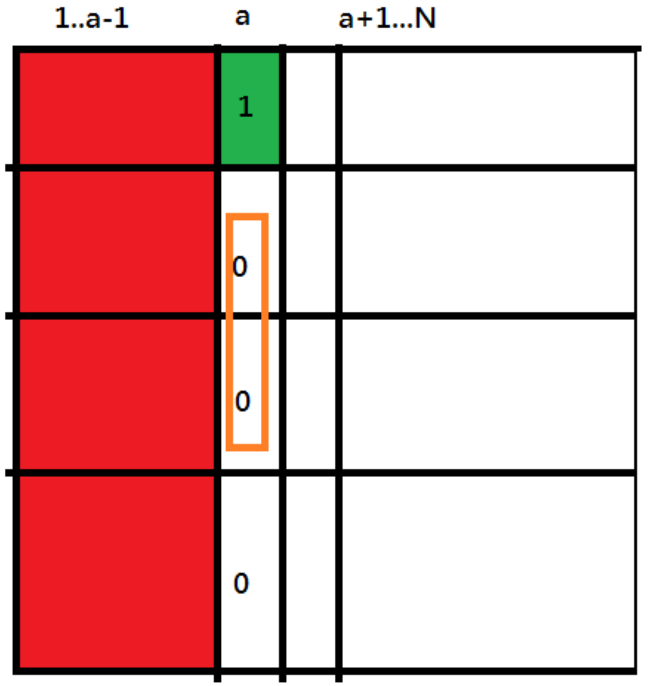


# Bitwise DP

or, we can put a vertial domino at the current column.

but after the bottomest “1” slot. Why?

--> dp[i][j] contributes to dp[i][j+2^k+2^(k+1)] as long as they are empty



# Bitwise DP

Time Complexity?
Number of state = O(2^M*N)
Transition per state = O(M)
--> O(MN*2^M)

What if the condition of “fully filled” is removed?
What if the 1*2 domino becomes 1*3?

# Tree DP

Tree: A graph so that there is only a single path from a vertex to any other vertex.

Many optimization problems in tree can be solved by tree DP.
Not a tree? --> biconnected component it sometimes may work. (Graph III)

# Tree DP

Two types of tree:

- Rooted: easy
- Unrooted: annoying

# Tree DP

Assume tree is rooted somewhere.

DP problem in tree --> uses nodes as state, and states' children as transition formula reference

# Tree DP

Easy example: Given a rooted tree of size N, each vertex is labeled a number, given Q queries, for each query find out the vertex that has the greatest number in some subtree.

# Tree DP

Let F(x) be the answer to the problem with subtree x.
F(x) = max(V_x, F(u): u is x’s children)

# Tree DP

Time complexity?

Each vertex will send a O(1) request to its children.
--> time complexity = O(sum of number of children in the tree) = O(N)

# Tree DP

Unrooted tree cases are more difficult but sometimes we can translate the unrooted case into rooted case (by assigning a random root to the tree and prove that optimal substructure exist)

# Tree DP

A little more difficult example:
Given a (unrooted) weighted tree of size N, find a set of paths so that no two paths uses a same vertex and the sum of weight of paths is maximal.

# Tree DP

Rephrase: Given the tree, find the set of edges so that a vertex is connected to at most 2 of the edges in the set and the sum of weight is maximal.

# Tree DP

If we root the tree (somewhere), then the answer of a subtree depends on the root, as well as the children of the root, and the number of edge the root can use.

F(u, x): maximal answer of subtree rooted at u and u can connect to at most x(1..2) vertex.

# Tree DP

How to calculate F(u, 1)?
if we fixed where u connects, let it be v, then
F(u, 1) = sum(F(t, 2): t is u's children + F(v,1) - F(v, 2) + w(u, v))

red part is fixed!
find a v such that the blue part is maximal.

# Tree DP

How to calculate F(u, 2)?
Similarly, if u connects to v and y, then
F(u, 2) = sum(F(t, 2): t is u's children + F(v,1) - F(v, 2) + w(u, v) + F(y, 1) - F(y, 2) + w(u, y))

find maximal and second maximal!

# Tree DP

Time complexity:
Number of state = O(N)
Each state is “referred” once, therefore sum of transition = O(N)
find maximal and second maximal --> O(1)
-->total = O(N)

# Tree DP

Sometimes story are not that happy.
We can try all vertex as root.

Drawback: Complexity increased by a factor of O(N)

# Tree DP

Problem: Given a tree, Let V(u) is the maximal distance between u and another vertex w. We are to find the minimal V(u) among all u.

Subproblem: How to find V(x) for a fixed x?

# Tree DP

Let F(a, x) be the maximal distance from x to one of its leafs in x's subtree (relative to a)

$F(a, x) = \max(F(a, t) + w(x, t): t \text{ is } x\text{'s children})$

$V(a) = F(a, a)$

# Tree DP

Back to main problem
Naive calculate V(u) for all u is O(N^2)
calculation of one value of V(u) is O(N).

But when we are doing V(t), prehaps we can save some information from the previous calculations

# Tree DP

Transition of root:
First we query the V value for a random root, say u
If we fix one of the u’s neighbour as new root, say v, what do we have beforehand?

# Tree DP

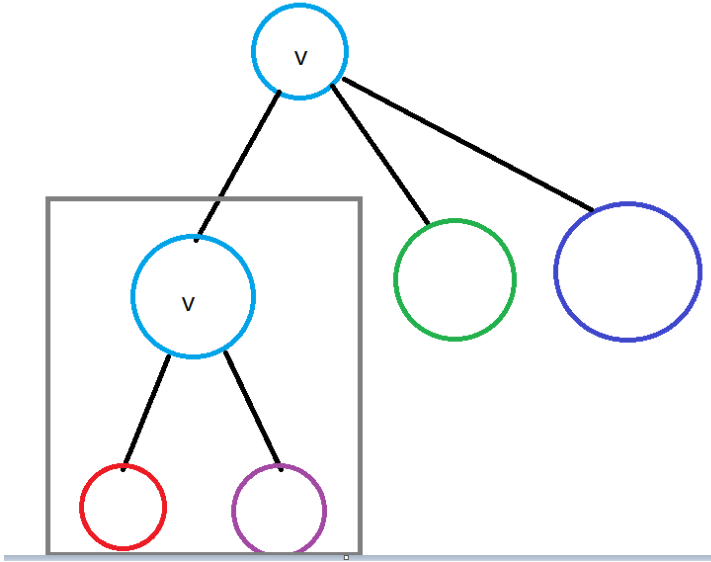
F(u, x) = max(F(u, t) + w(x, t): t is x's children)

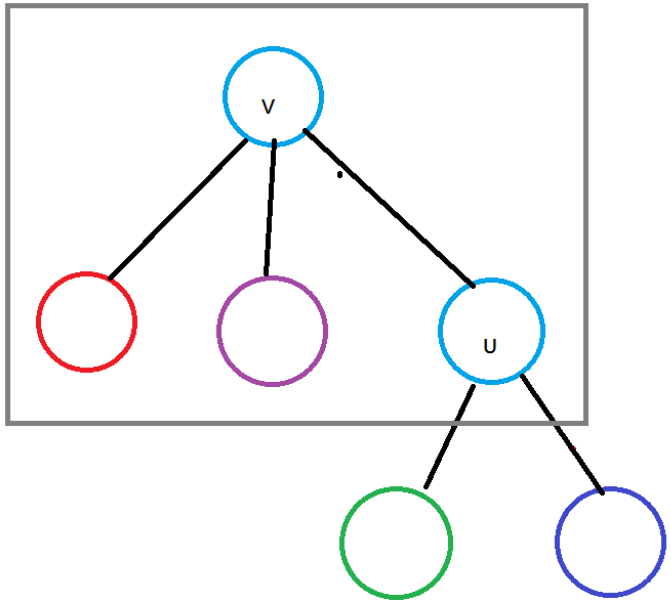
F(u, v) = max(F(u, t) + w(v, t): t is v's children)

F(v, v) = max(F(v, t) + w(v, t): t is v's children)

# Tree DP

They turn out to be very similar...



# Tree DP

$F(u, v) = \max(F(u, t) + w(v, t): t \text{ is } v\text{'s children})$
$F(v, v) = \max(F(u, v), F(v, u) + w(v, u))$

How to calculate red terms in O(1)?
Divide into two cases:

# Tree DP

- F(u, u) does not depends on v
  --> F(v, u) = F(u, u)
- F(u, u) depends on v
  --> F(v, u) = second_maxial_of_F(u, u)
  as we are forbidden to use (u, v) path
  in calculation of F(v, u)

--> rotation of roots can be done in O(1)

# Tree DP

Challenge:
Given a unrooted tree of size N, each vertex is labeled a number, given Q queries, for each query find out the vertex that has the greatest number in some subtree relative to another neighbour vertex.

# More to read

https://www.topcoder.com/community/data-science/data-science-tutorials/dynamic-programming-from-novice-to-advanced/

http://codeforces.com/blog/entry/8219