HKOI 2015/16 Solution
Senior Q2 (Robos' Feast)

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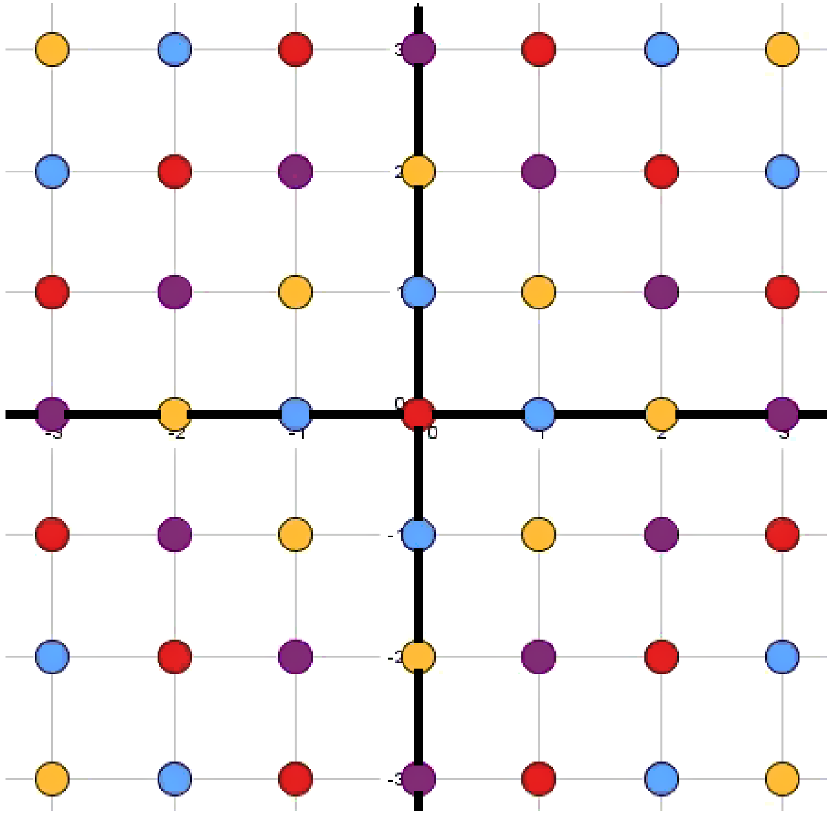
23/1/2016

Task Description

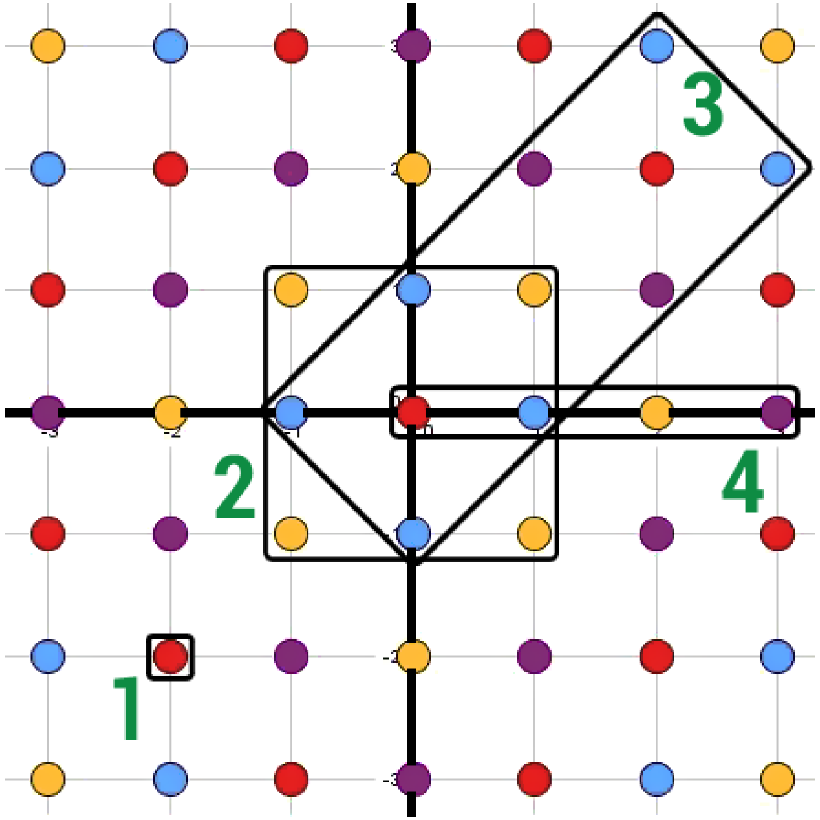
- Cartesian plane
- All integer points (x,y) are colored according to (|x|+|y|) mod 4

- Task: Find the total value of points inside a given rectangle



Task Description

- Type 1: the edges of the rectangle are axis-parallel (1, 2, 4)
- Type 2: the edges of the rectangle are 45 degrees to the axes (3)



Task Description

- For all test cases, 1 <= N <= 10000; -10^8 <= coordinates <= 10^8

- Subtask 1 (20 points): $T_R = T_B = T_O = T_P$
- Subtask 2 (35 points): 1 <= N <= 100; -100 <= coordinates <= 100
- Subtask 3 (45 points): No additional constraints

- Within a subtask, if you only handle one type of action correctly for each case, you get 80% of the score of that subtask

Motivation

- In HKOI training team, a trainer, whom we call RB, likes to "eat salt"
- In the original task setting, the objects on the integer points are salt instead of oil bottles
- Colors are Red, Blue, Orange, Purple ☺

(Source: http://patienttalk.org/wp-content/uploads/2013/03/Salt-cartoon-2.png)

Statistics

Attempts	Max	Mean	Std Dev	Subtasks		
42	80	24.095	20.859	20: 0 16: 15	35: 0 28: 25	45: 0 36: 2

- In the contest, <u>no one</u> was able to handle Type 2 correctly ☹

Presentation Flow

- Subtasks 1 and 2, Type 1
- Subtask 3, Type 1

- Subtasks 1 and 2, Type 2
- Subtask 3, Type 2

Type 1

- Axis-parallel rectangles
- First, swap coordinates so that $x_1 <= x_2$ and $y_1 <= y_2$

Subtask 1 (all points have the same value)

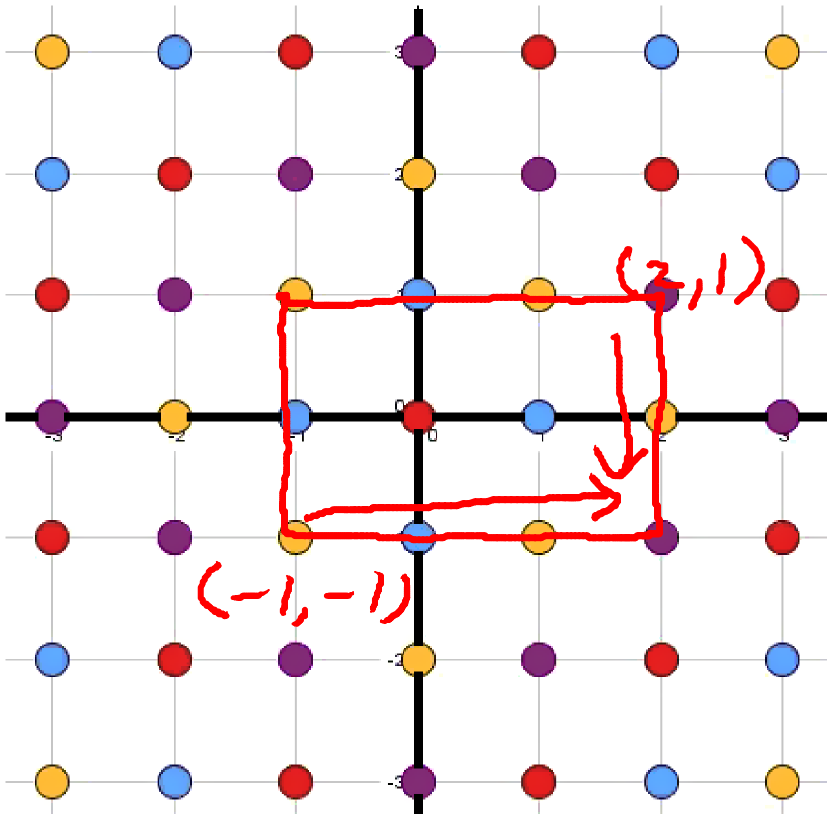
- $T_R = T_B = T_O = T_P$
- How to make use of this constraint?

Subtask 1 (all points have the same value)

- Answer = P * T_R
  - P: number of points inside the rectangle

- $P = (x_2 - x_1 + 1) * (y_2 - y_1 + 1)$

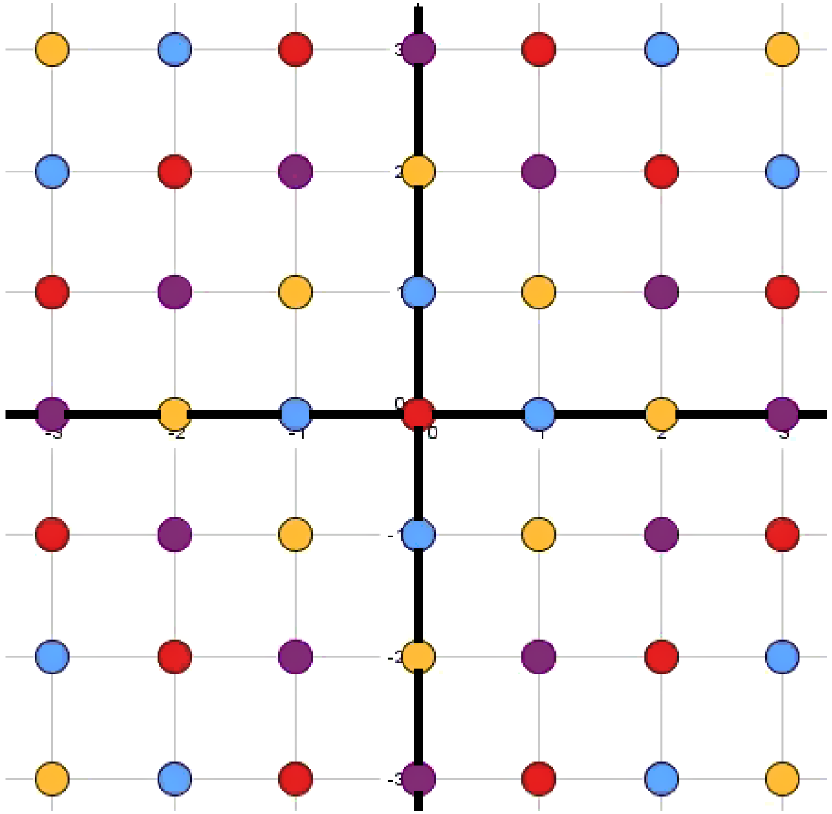
$[2-(-1)+1]*[1-(-1)+1]$



Subtask 2 (small coordinates, few Robos)

- Brute force

- for i from x_1 to x_2
- for j from y_1 to y_2
- add value of point (i, j) to answer

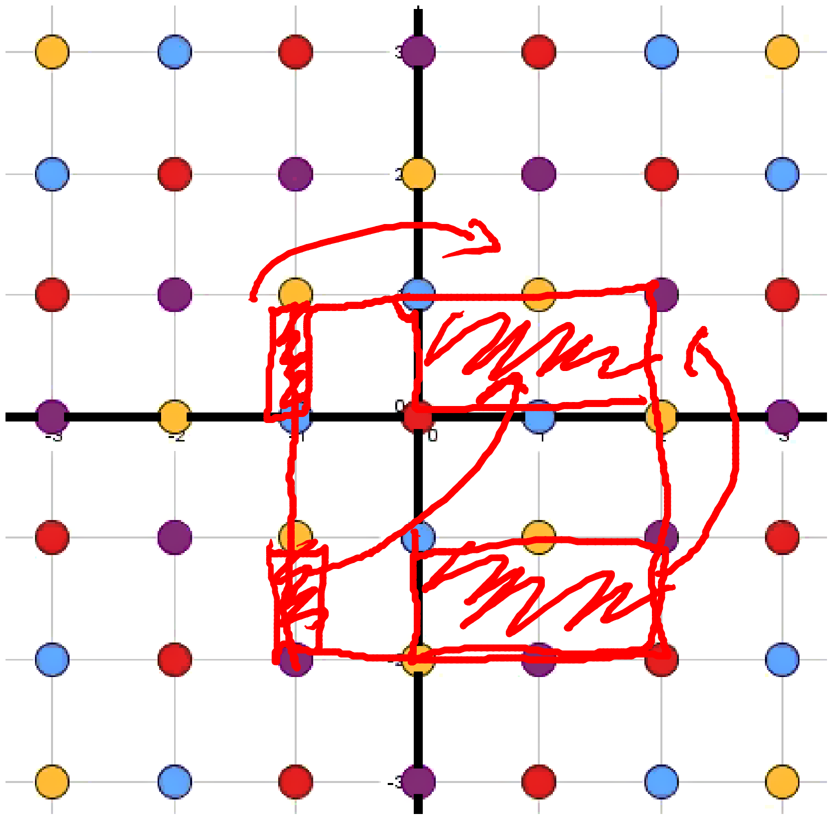


Subtask 3

- Clever counting
- Use “tricks” to make counting easier

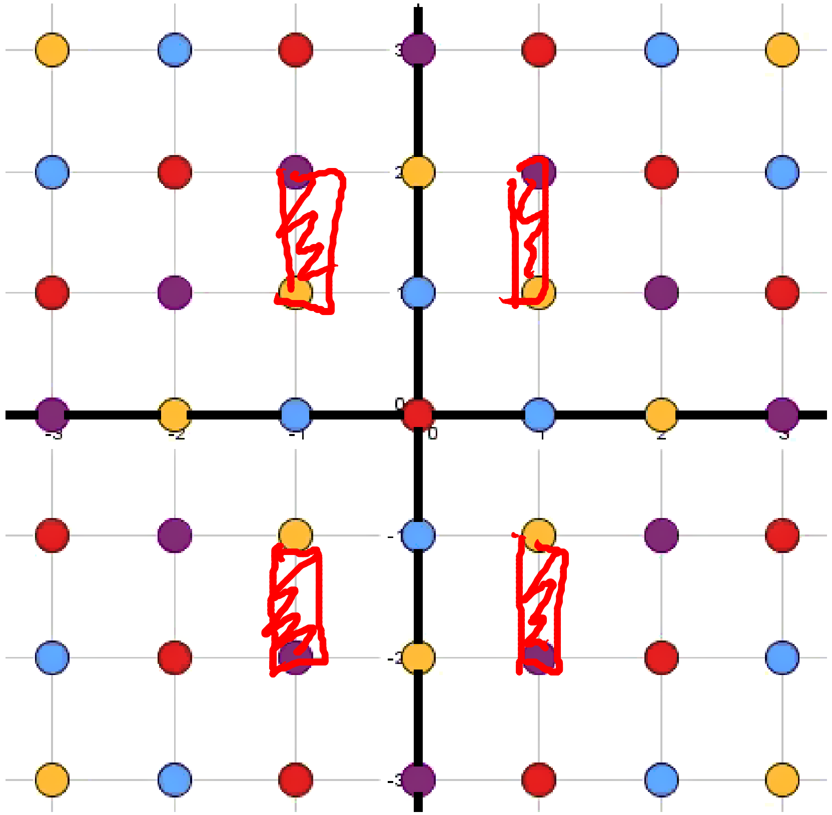
Subtask 3

- Trick 1: Break given rectangle into pieces



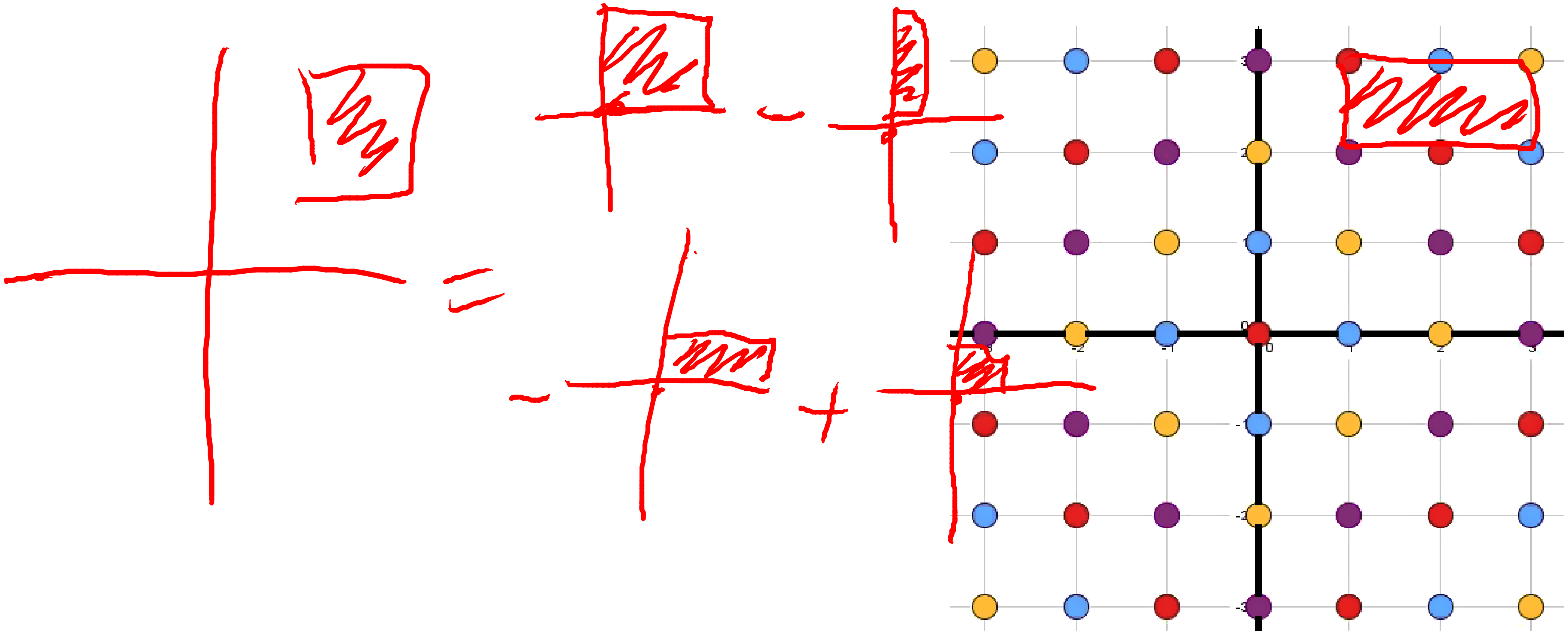
Subtask 3

- Trick 2: Use symmetry



Subtask 3

- Trick 3: Use inclusion-exclusion



Subtask 3

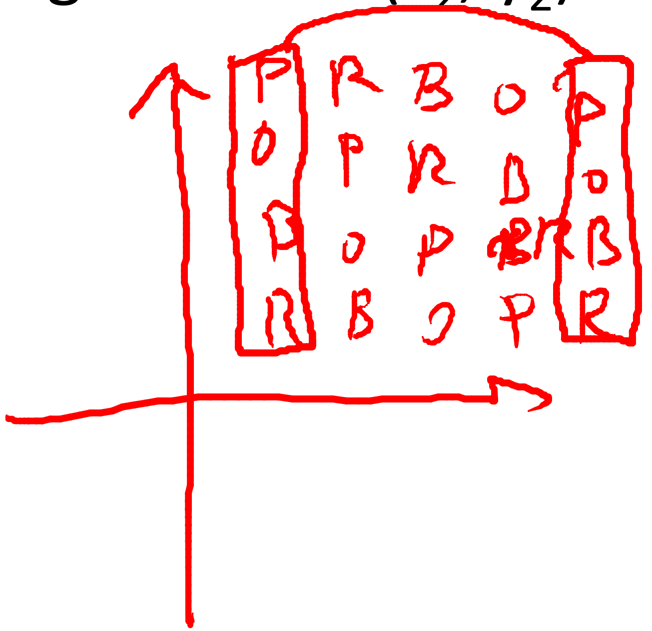
- Current (simplified) setting:

- $(x_1, y_1) = (0, 0)$
- $x_2, y_2 >= 0$

Subtask 3

- Observation

- Let $S(k)$ be the sum of values of points on the line $x = k$ and inside a rectangle with bottom-left corner $(0, 0)$ and top-right corner (x_2, y_2)
- Then, if
  - 1) $k_1 \% 4 = k_2 \% 4$, and
  - 2) Both k_1 and k_2 are in $[0, x_2]$



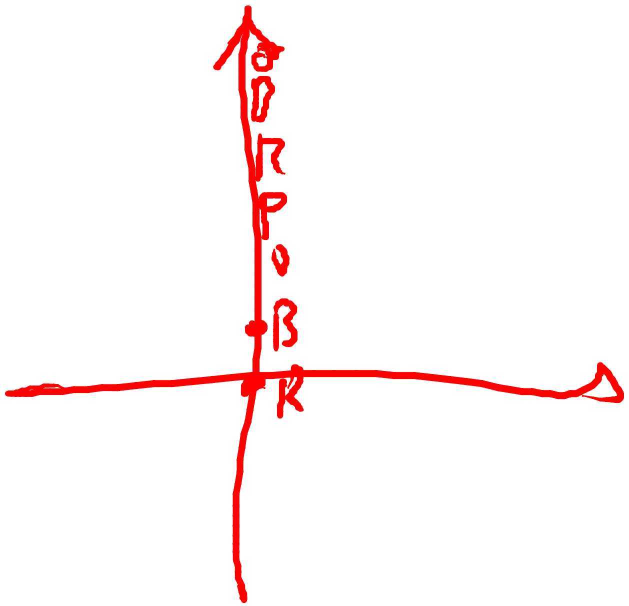
- We have $S(k_1) = S(k_2)$

Subtask 3

- Meaning: the “column sum” of points are *periodic* with period = 4
- Therefore, we only need to calculate S(0), S(1), S(2), and S(3)

Subtask 3

- Consider S(0), i.e. sum of values of points (0, 0), (0, 1), …, (0, y_2)
- **RBOP**RBOP**RBOP**RBOP**RBOP**RBOPR…
- Again, we find it *periodic* ☺



Subtask 3

- RBOPRBOPRBOPRBOPRBOPRBOPR…

- Number of ‘R’s = $(y_2 + 4) / 4$
- Number of ‘B’s = $(y_2 + 3) / 4$
- Number of ‘O’s = $(y_2 + 2) / 4$
- Number of ‘P’s = $(y_2 + 1) / 4$

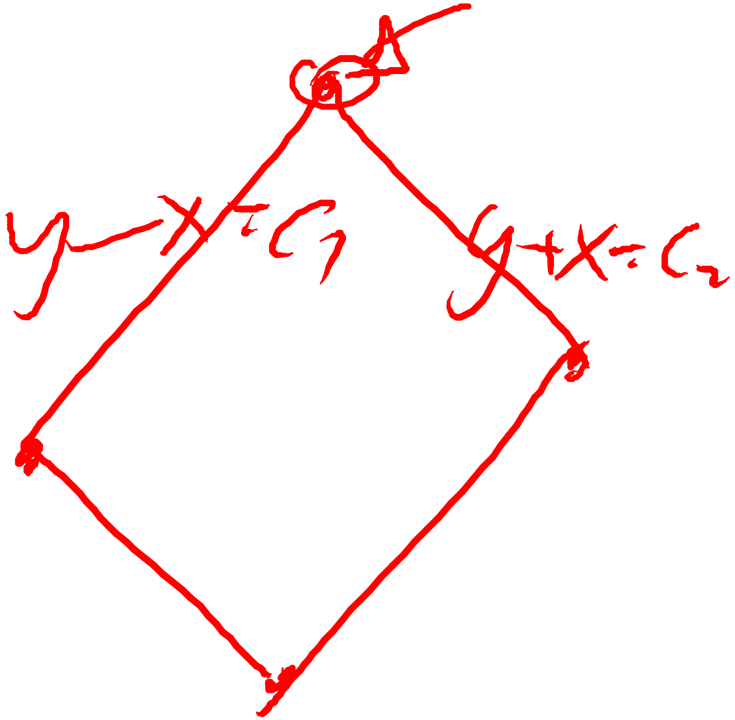
- DONE! (S(1), S(2), S(3): very similar)

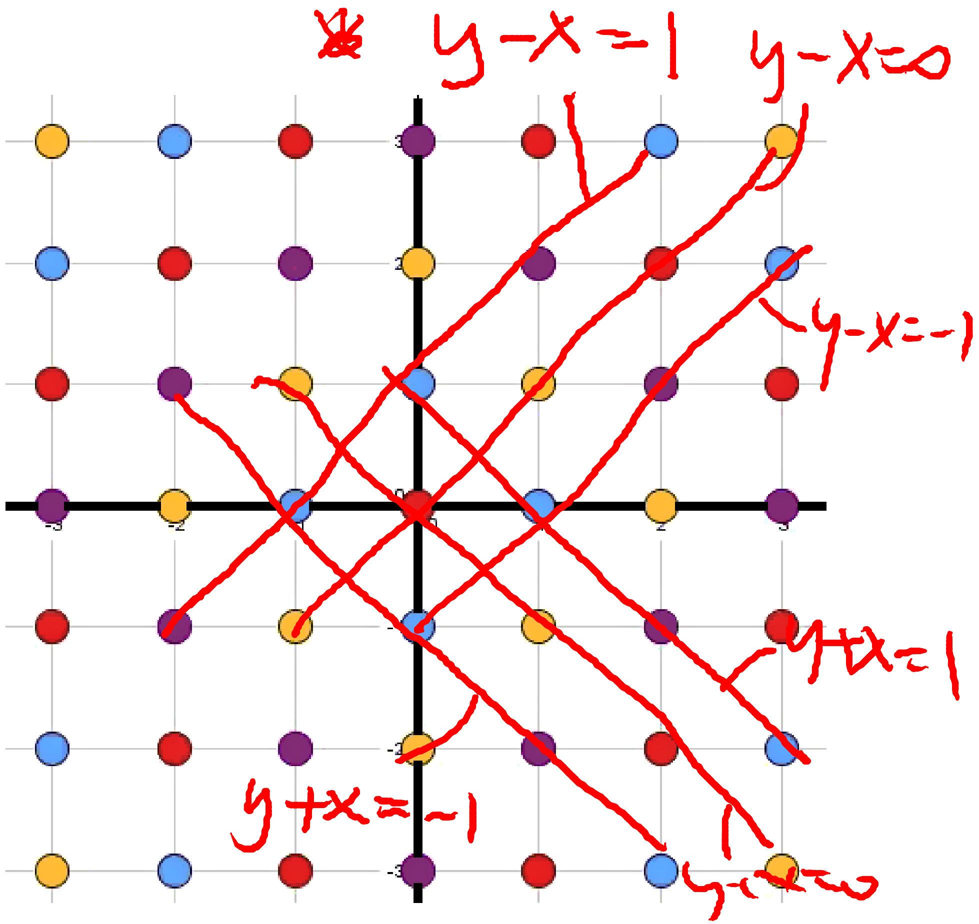
Type 2

- Edges of rectangles are 45 degrees to the axes
- “Ugly” setting ☹

Basic Geometry

- Equation of lines 45 degrees to the axes:
- 1) y = x + c, or
- 2) y = -x + c



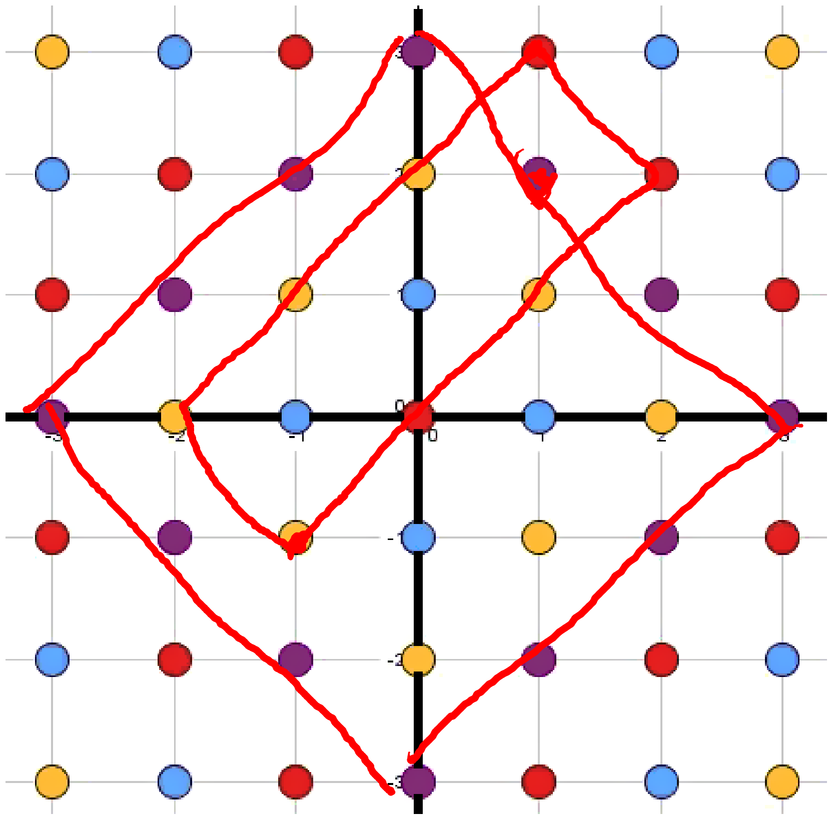


Subtask 1 (all points have the same value)

- Answer = P * T_R
  - P: number of points inside the rectangle

- Let us try some examples

- (-1, -1), (1, 3) → 11
- (0, -3), (0, 3) → 625

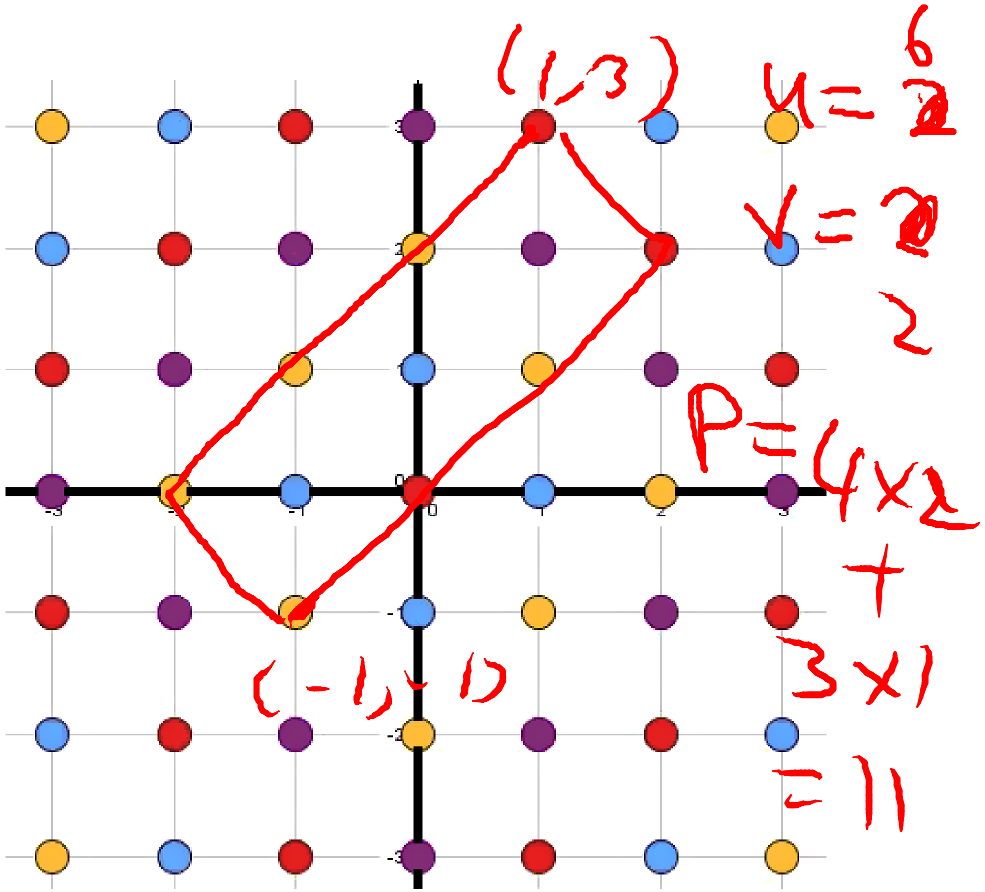


Subtask 1 (all points have the same value)

- Answer = P * T_R
  - P: number of points inside the rectangle

- Formula found by several contestants:

- Let $u = |(x_1-y_1) - (x_2-y_2)|$
- Let $v = |(x_1+y_1) - (x_2+y_2)|$
- $P = (u/2+1)*(v/2+1) + (u/2)*(v/2)$

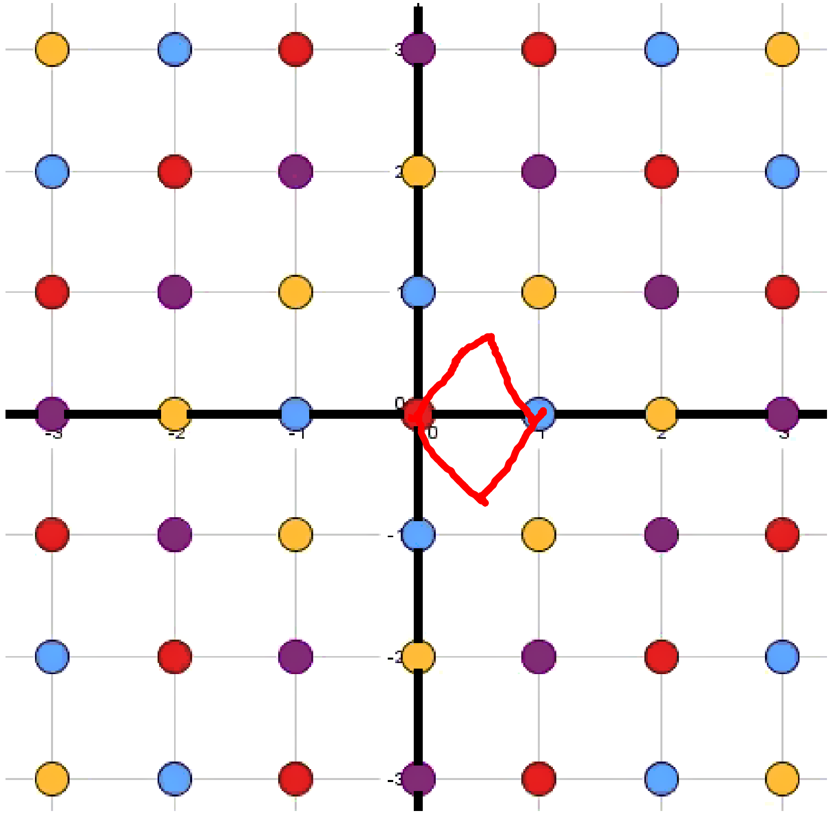


Subtask 1 (all points have the same value)

- Let $u = |(x_1-y_1) - (x_2-y_2)|$
- Let $v = |(x_1+y_1) - (x_2+y_2)|$
- $P = (u/2+1)*(v/2+1) + (u/2)*(v/2)$

- Q: Why is the formula (sometimes) wrong?
- A: The two unspecified corners may not have integer coordinates.

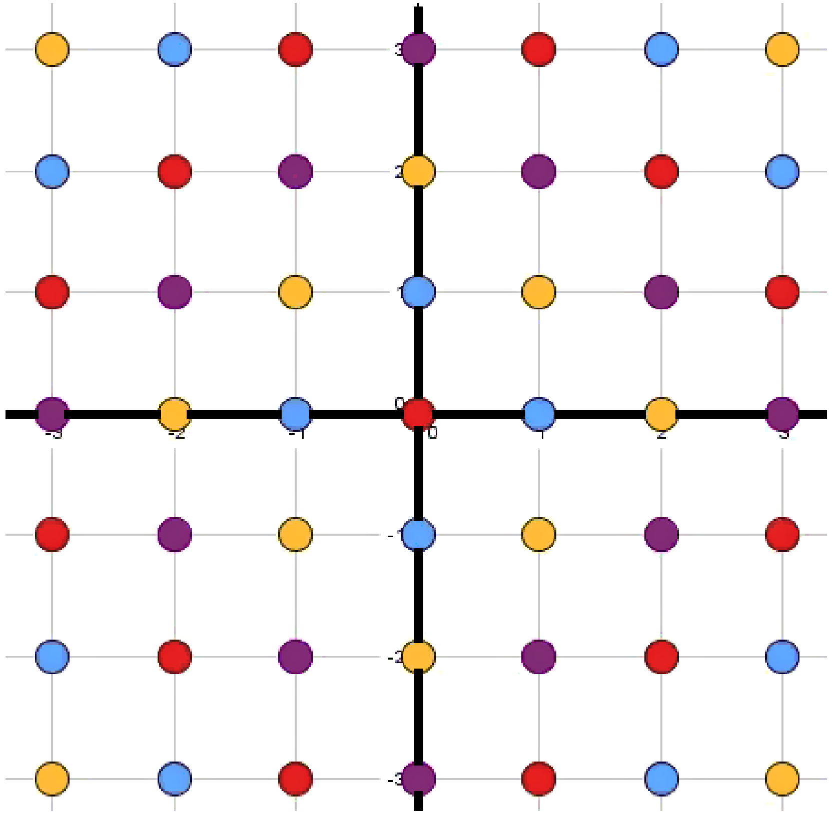
- e.g. (0, 0), (0, 1)



Subtask 1 (all points have the same value)

- Q: When is the formula correct?
- A: When the two unspecified corners have integer coordinates.

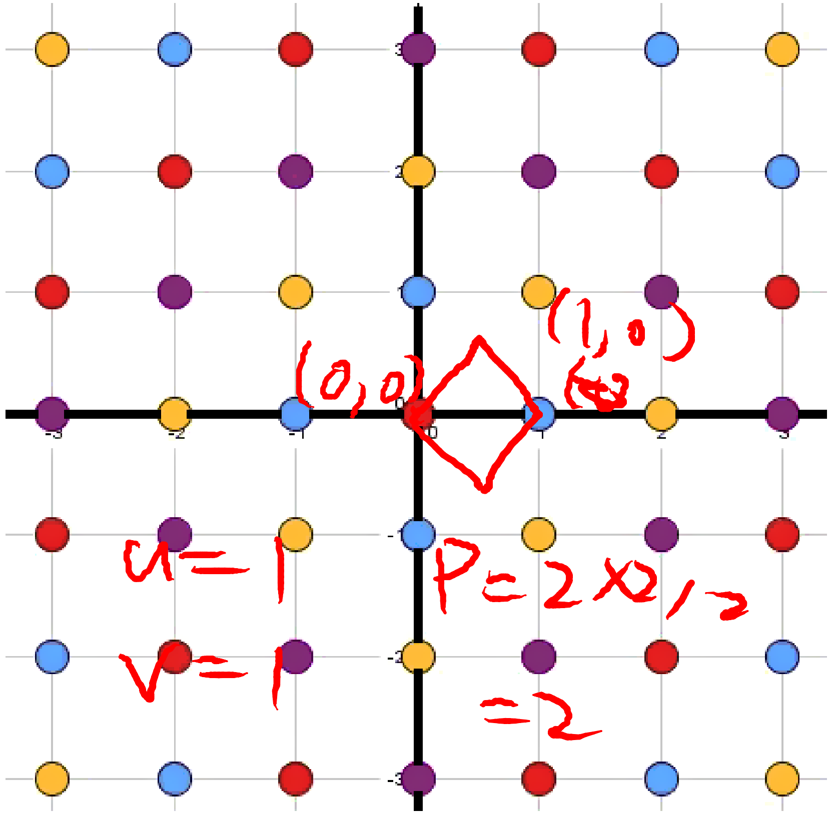
- Q: What does this mean?
- A: This means $(x_1+y_1+x_2+y_2)$ is even.



Subtask 1 (all points have the same value)

- Let $u = |(x_1 - y_1) - (x_2 - y_2)|$
- Let $v = |(x_1 + y_1) - (x_2 + y_2)|$

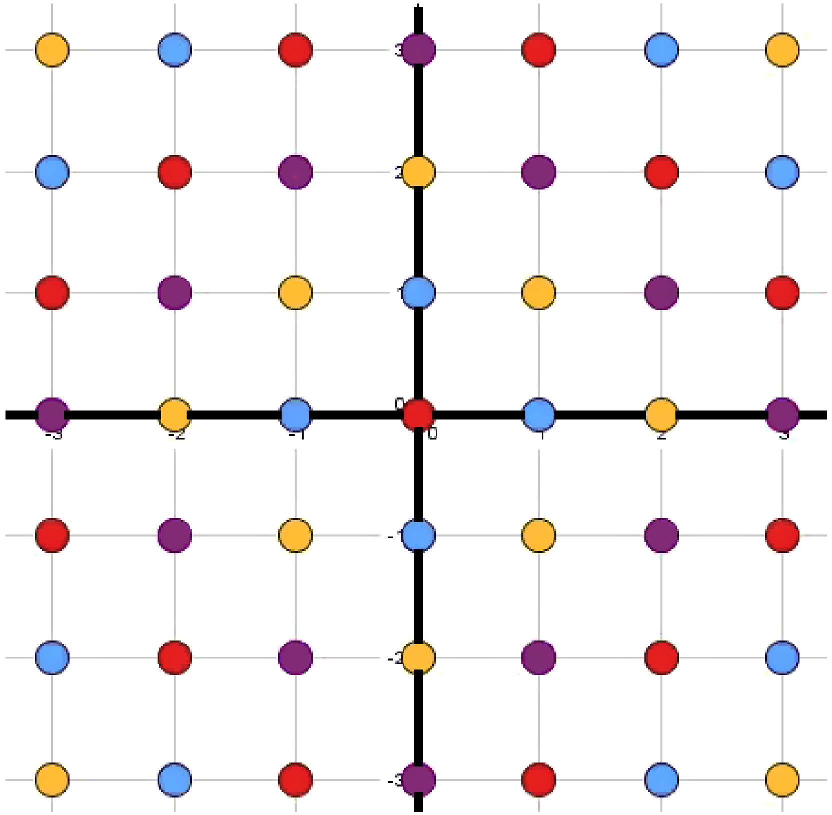
- If $(x_1 + y_1 + x_2 + y_2)$ is even
  - Use $P = (u/2 + 1) * (v/2 + 1) + (u/2) * (v/2)$
- Else
  - $P = ((u+1) * (v+1))/2$



Subtask 1 (all points have the same value)

- Let $u = |(x_1 - y_1) - (x_2 - y_2)|$
- Let $v = |(x_1 + y_1) - (x_2 + y_2)|$

- Or, just $P = ((u+1) * (v+1) + 1)/2$



Subtask 2 (small coordinates, few Robos)

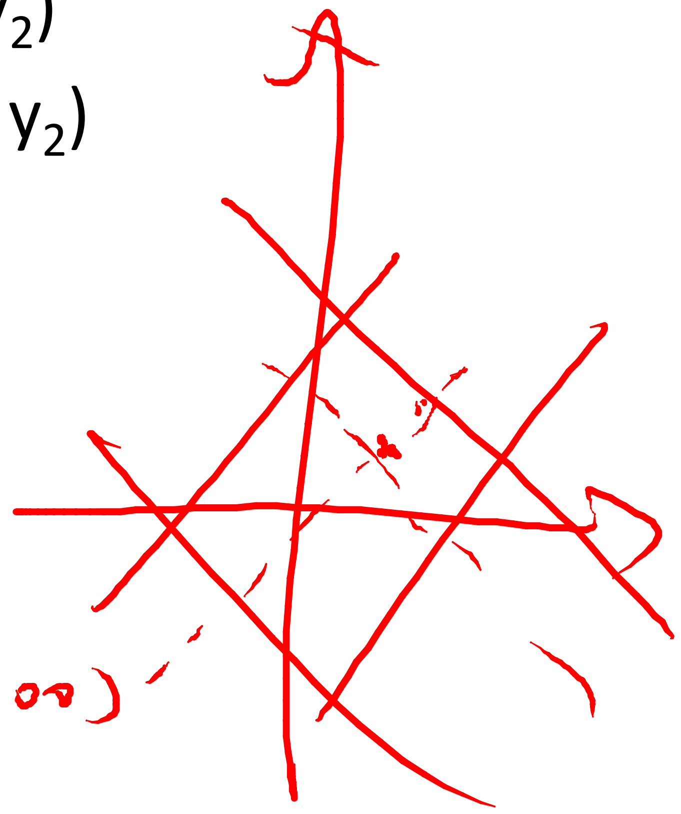
- Brute force

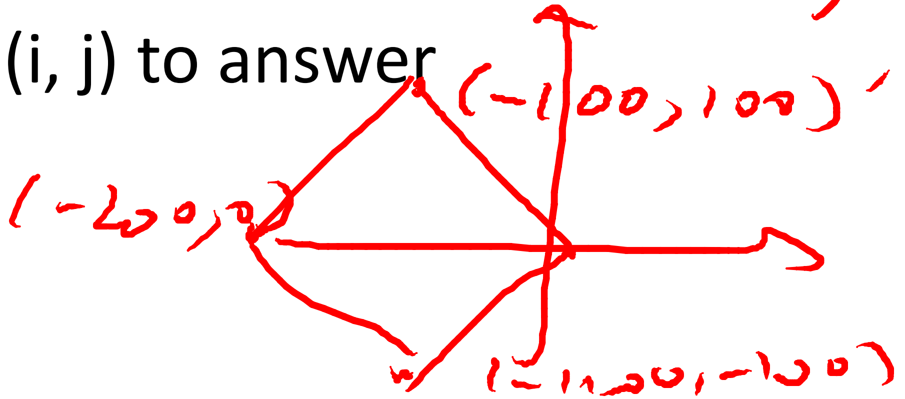
- For each point with “small” coordinates, determine whether it is inside the rectangle

Subtask 2 (small coordinates, few Robos)

- Let la = min($x_1 - y_1$, $x_2 - y_2$),   lb = min($x_1 + y_1$, $x_2 + y_2$)
- Let ua = max($x_1 - y_1$, $x_2 - y_2$), ub = max($x_1 + y_1$, $x_2 + y_2$)

- for i from -200 to 200
-     for j from -200 to 200
-         if (la <= (i – j) <= lb) and (ua <= (i + j) <= ub)
-             add value of point (i, j) to answer





Geometric Transformation

- Map all points (x, y) in the rectangle to (x-y, x+y)

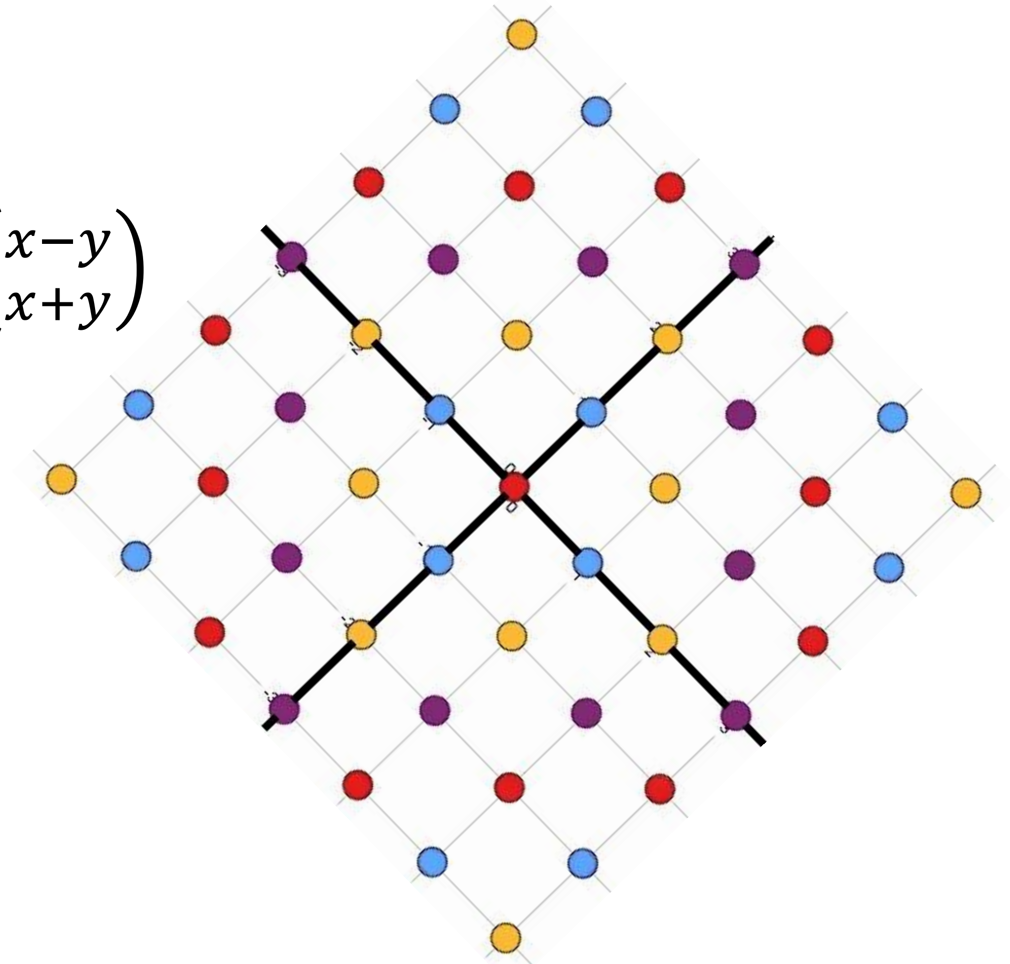
- Geometric meaning:
- 1) Rotate the rectangle by 45 degrees about the origin, then
- 2) Enlarge the rectangle by $\sqrt{2}$ about the origin

Geometric Transformation

- For those who know matrices,
- $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \sqrt{2} \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$

$\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$



Subtask 3

- 1. Apply said transformation
- 2. Use tricks (breaking into pieces, symmetry, inclusion-exclusion)
- 3. Calculate the answer in a simplified setting

Subtask 3

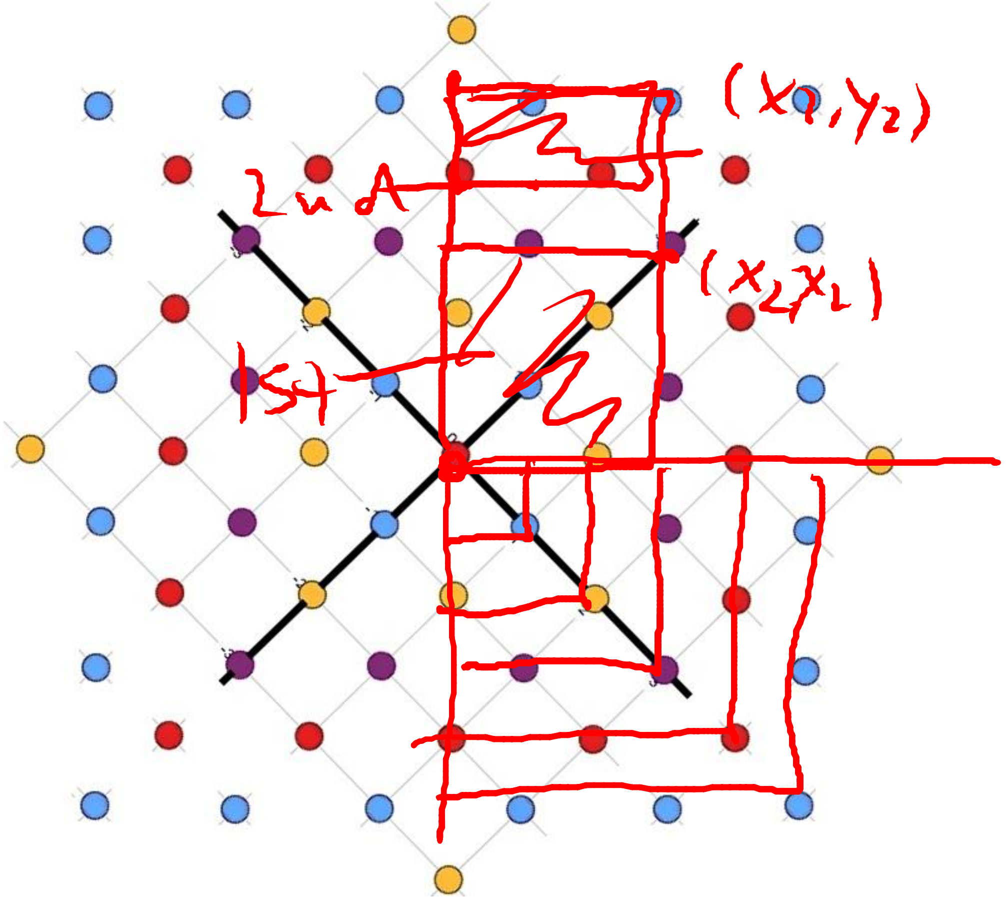
- Current (simplified) setting:

- $(x_1, y_1) = (0, 0)$
- $y_2 >= x_2 >= 0$ (If not, just swap x_2 and y_2)

Subtask 3

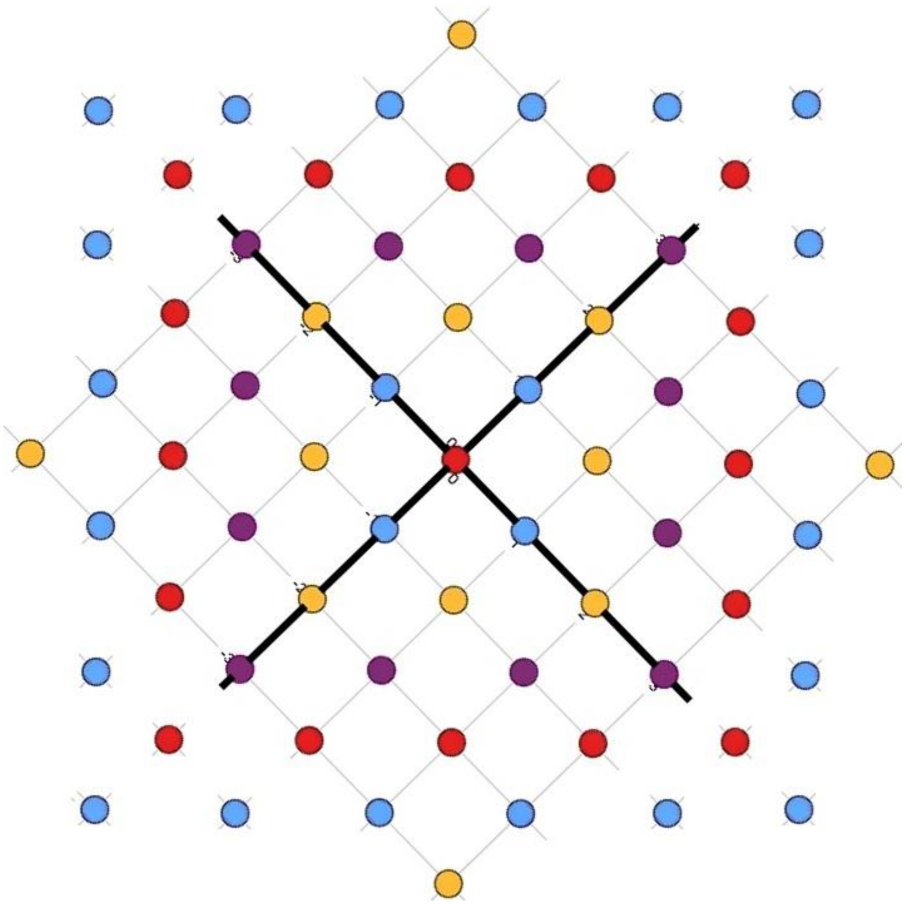
- Break the rectangle into two parts

- First part: 1R, 1B, 3O, 3P, 5R, 5B, …
- Second part:
  - If x_2 is even, …
  - If x_2 is odd, …



Subtask 3

- First part: 1R, 1B, 3O, 3P, 5R, 5B, …
- Consider R only: 1R, 5R, 9R, …
- Arithmetic sequence
- $1 + 5 + \ldots + (4n + 1) = (n + 1) * (2n + 1)$
- $3 + 7 + \ldots + (4n + 3) = (n + 1) * (2n + 3)$

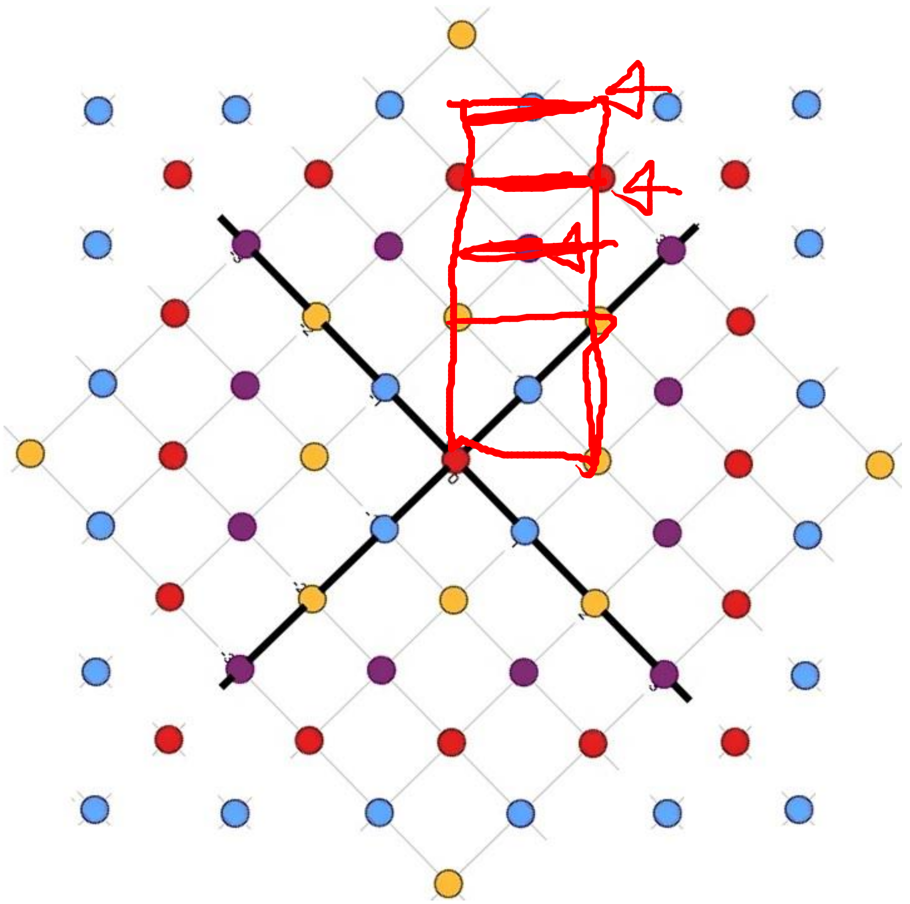


Subtask 3

- Second part:
  - If x_2 is even, …

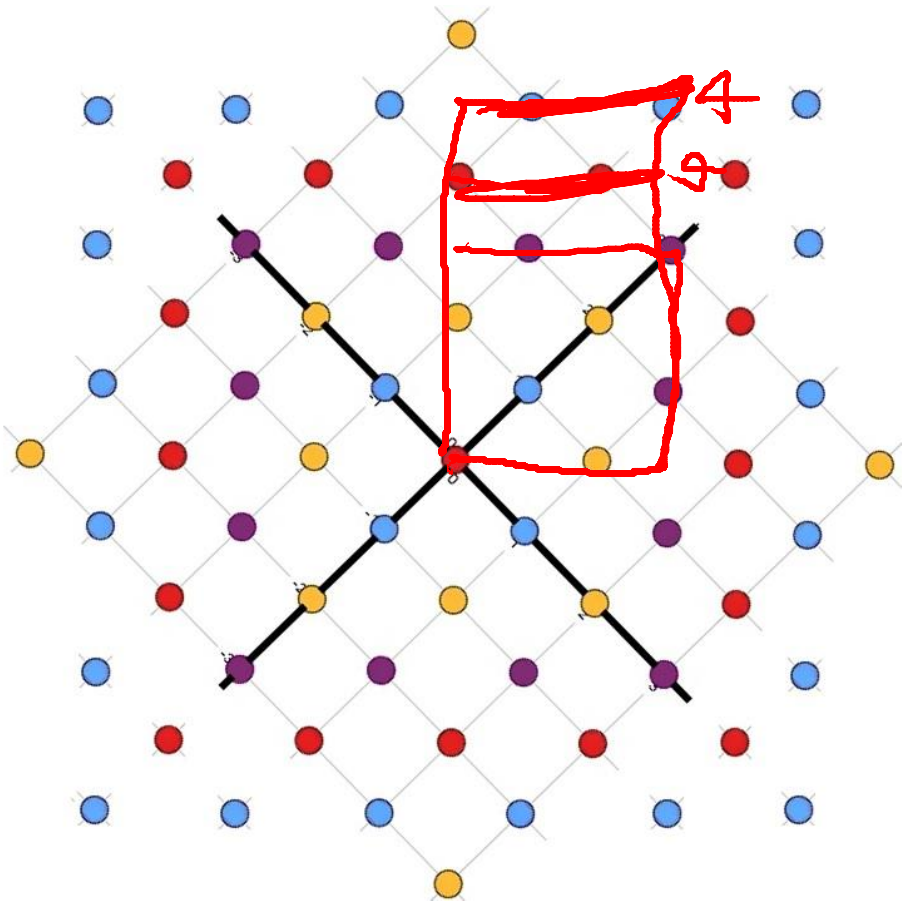
- We may have 3P, 4R, 3B, 4O, 3P, 4R, …

- For arbitrary x_2,
  - Replace ‘3’ with $x_2/2$
  - Replace ‘4’ with $x_2/2 + 1$



Subtask 3

- Second part:
  - If x_2 is odd, …
- We may have 3P, 3R, 3B, 3O, 3P, 3R, …
- For arbitrary x_2,
  - Replace ‘3’ with $(x_2 + 1)/2$



Summary

- Subtask 1: find fancy formula to calculate number of points included
- Subtask 2: try all “possible” points
- Subtask 3: use tricks to simplify task, then do O(1) calculation

- I leave the implementation details to you ☺
  - With good coding skills, you may solve this task within 100 lines of code

Thank you

- Any questions?