Father's Will

Problem Setter : Yuen Chak Fai

Test Data: LMH + Gagguy

Problem - Background

- N houses
  - Base price H_i
  - $H_i \leq H_{i+1}, (1 \leq i < N)$
- Price correlate with phase cycle
  - Phase 1 (Trough) : $1 \times H_i$
  - Phase 2 (Recovery) : $2 \times H_i$
  - Phase 3 (Peak) : $3 \times H_i$
  - Phase 4 (Recession) : $2 \times H_i$

Problem - Background

- Initial Condition
  - At Phase 1 - Trough
  - Money - M
  - Cheapest House – H_1
- Operation
  - One house at any time
  - Sell & Buy
  - Immediate Transaction

Problem - Objective

- Buying King's Garden – H_N

- Calculate the earliest time

Problem - Sample

- Sample Case 1

N	M	H_i	Output
3	4	2 5 10	4

- Sample Case 2

N	M	H_i	Output
2	96	1 100	Impossible

Problem – Sample 1 Explanation

N	M	H_i	Output
3	4	2 5 10	4

Year	House 1	House 2	House 3	Owning	Cash
Initial	2	5	10	House 1	4
0	2	5	10	House 2	1
1	4	10	20	House 2	1
2	6	15	30	House 1	10
3	4	10	20	House 1	10
4	2	5	10	House 3 *	2

Problem – Sample 2 Explanation

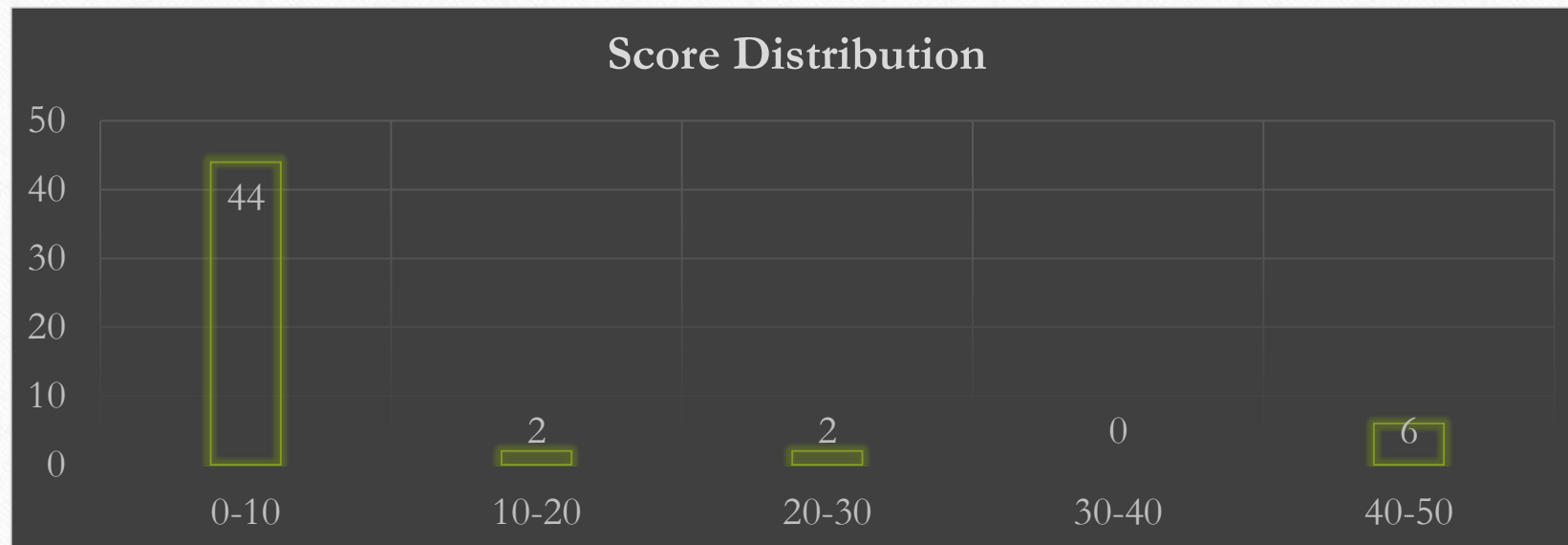| N	M	H_i	Output
2	96	1 100	Impossible

- No possible transaction
- Necessary Sufficient Condition for Impossible?

Constraint

Subtask	Max Point	N	H_i	M
1	30	$2 \leq N \leq 100$	$1 \leq H_i \leq 10000$	$1 \leq M \leq 10000$
2	15	$2 \leq N \leq 1000$	$1 \leq H_i \leq 10^6$	$1 \leq M \leq 10^6$
3	15	$2 \leq N \leq 1000$	$1 \leq H_i \leq 10^{14}$	$1 \leq M \leq 10^{14}$
4	40	$2 \leq N \leq 100000$	$1 \leq H_i \leq 10^{14}$	$1 \leq M \leq 10^{14}$

Statistics

Attempts	Max	#Max	Min	Mean (Att.)	St Dev (Att.)
54	45	2	0	6.96	14.19

Score Distribution

Problem Analysis

- Key: Finding the best general strategies

- Optimize!

- Trivial Observation

  - Best phase to buy a better house – Phase 1

  - Best phase to switch to a worse house – Phase 3

Problem Analysis - Observation

- Buying
  - What to buy?

Solution – General Solution

- Phase 1 – Buy the best you could afford

- Phase 2 – Do nothing

- Phase 3 – Buy the cheapest

- Phase 4 – Do nothing

- Loop…

  - Special Handle in the end

Proof of algorithm optimality

Lets restrict our goal to earning money instead of buying house n. We will show that our algorithm is optimal to earn money by induction on time.

Claim 1: when we switch to a better house, we should do it on phase 1 (Though).

Proof. Assume we sell house i and buy house j $(i < j)$ at some other phases, we will show that it is better switch it to phase 1. First notice that we need at least $c(h_j - h_i)$ cash to perform such operation where $c \in \{1, 2, 3\}$ depend on the phase. Consider the most recent phase 1 before the operation, notice that we always have enough cash to perform the same operation at phase 1. As we were able to perform the operation earlier with even more money leftover, we will be able to perform any subsequent operations afterwards provide that the original plan is able to perform. Therefore it is always better to switch to a better house only at phase 1. □

Claim 2: when we switch to a lesser house, we should do it on phase 3 (Peak).

Proof. Assume we perform such operation on phase 2 (Recovery), we will show that it is better to wait until the next year which is phase 3. It is always possible to do so as it does not require any extra cash. Also, if we perform some operations in phase 3 originally it is still possible to do so as we have more cash leftover.

Similarly, if we perform such operation phase 4 (Recession), we should perform it one year earlier which again must be possible, having more cash leftover and we have shown that earlier is always better (provided that other parameters did not get worse). □

Combining the two claims, we can conclude that we only perform operations on phase 1 and 3.

Claim 3: when we switch to a better house, we should switch to the best one we can afford.

Proof. Let house i be the current house we have. Consider house j, k $(j < k)$, and the following operations:

Sell house i and buy house j at phase $1 \rightarrow$ (after some time) $\rightarrow$ Sell house j and buy house t

in this process our change of cash equals $(h_i - h_j) + c(h_j - h_t) = h_i - ch_t + (c-1)h_j$ (where $c = 1$ or 3 depends on the phase). Notice that if we bought house k and switch to house t at the same time as the original plan, our change of cash equals $h_i - h_t + (c-1)h_k$. Since $h_k > h_j$, it is better. □

Claim 4: when we switch to a lesser house, we should switch to house 1.

Proof. Similar to above, consider the following operations where $1 \le j < i$:

Sell house i and buy house j at phase $3 \rightarrow$ (after some time) $\rightarrow$ Sell house j and buy house t

in this process our change of cash equals $3(h_i - h_j) + c(h_j - h_t) = 3h_i - ch_t - (3-c)h_j$ (where $c = 1$ or 3 depends on the phase). Therefore, we would like h_j to be as small as possible. □

Finally, notice that it is not optimal to do two consecutive "switch to a better house" or "switch to a lesser house", we should alternate between getting a better and lesser house. To see that we should do it on **every** cycles, notice that we will earn $2(h_i - h_1) > 0$ by doing so, therefore it is better than doing nothing.

1

Solution - Subtask 1

Subtask	Max Point	N	H_i	M
1	30	$2 \leq N \leq 100$	$1 \leq H_i \leq 10000$	$1 \leq M \leq 10000$

- Simulate year by year
- Buy the best you could afford
  - Loop through N houses
- Time Complexity?

Solution - Subtask 1

N	M	H_i	Output
3	1	1 2 100000	199996

- Each cycle
  - Can earn $2
  - Need to loop through N houses
- $O(N \times H_N)$

Solution – Optimization

- Avoid repetitive work
  - Fast Forward
  - Shifting Pointer

Solution - Subtask 2 & 3 (Fast Forward)

Subtask	Max Point	N	H_i	M
2	15	$2 \leq N \leq 1000$	$1 \leq H_i \leq 10^6$	$1 \leq M \leq 10^6$
3	15	$2 \leq N \leq 1000$	$1 \leq H_i \leq 10^{14}$	$1 \leq M \leq 10^{14}$

N	M	H_i	Output
3	1	1 2 100000	199996

- Listing the houses we owned in i_{th} year

0	1	2	3	4	5	6	7	8	9	10	...	199996
2	2	1	1	2	2	1	1	2	2	1	...	3

Solution - Subtask 2 & 3 (Fast Forward)

- Calculate years needed to buy the greatest house we have ever own - H_x

  - $\left\lceil \dfrac{How\ much\ money\ we\ still\ need\ to\ buy\ H_x}{Money\ we\ earned\ in\ each\ cycle} \right\rceil \times Period$

- $Period = 4$

- How much money we still need to buy H_x

  - Buy in phase 1

  - $H_x - cash$

- Money we earned in each cycle

  - Sell in phase 2

  - $2 \times (H_{x-1} - H_1)$

Solution - Subtask 2 & 3 (Fast Forward)

- Finding best house we could buy
  - $O(N)$
- Max. number of intermediate houses
  - $O(N)$
- Time Complexity
  - $O(N \times N) = O(N^2)$

Solution - Subtask 4 (Fast Forward + Shifting Pointer)

- Finding best house we could buy - H_y
  - If we own H_x before, $y > x$
  - i.e. No need to reset pointer
  - Shifting Pointer!

Solution - Subtask 4 (Fast Forward +Shifting Pointer)

- Finding best house we could buy
  - $O(N)$
- Max. number of intermediate houses
  - $O(N)$
- Time Complexity
  - $O(N + N) = O(N)$

Solution – Tricky Cases

- Duplicate Values

- Extreme Answer Values

N	M	H_i	Output
3	1	1 2 10000000000000	19999999999996

- Answer not multiple of 4

  - Check if buy it one year earlier

N	M	H_i	Output
3	1	1 10 11	3

Solution – Pseudo Code

1. $H_{N+1} = \infty$
2. cash = M + H_1
3. REPEAT
4.   WHILE $H_y \leq cash$
5.   y = y + 1
6.   END WHILE
7.   IF $y > N$
8.   IF $cash + H_1 \geq H_N \times 2$ AND $years > 0$
9.   years = years - 1
10.   OUTPUT years
11.   EXIT
12.   END IF
13.   END IF
14.   diff = H_y - cash
15.   earned = 2 * $(H_{x-1} - H_1)$
16.   ncycle = $\left\lceil \frac{diff}{earned} \right\rceil$
17.   years = year + 4 * ncycle
18.   cash = cash + ncycle * earned
19. END REPEAT

Questions?