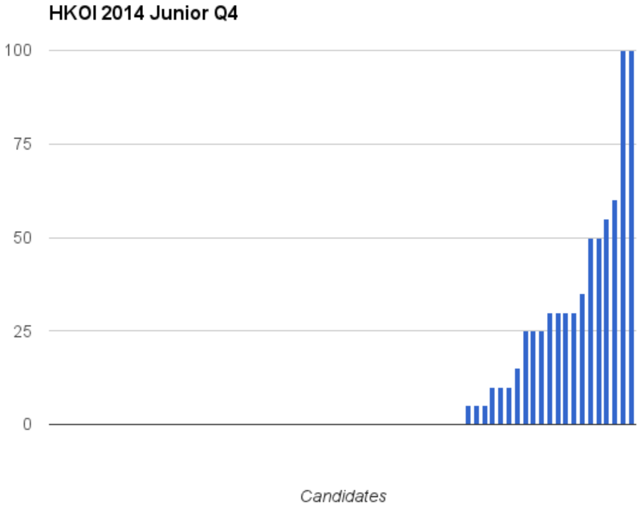
HKOI Final 2014 Solution
Junior Q4: Fair Santa Claus

January 11, 2014

Problem statement

- Given 3 integers n, a, b, promised $a + b = n$.
- Partition $[n] = \{1, 2, 3, \ldots, n-1, n\}$ into two sets α and β with size a and b respectively.
- Let the sum of elements in the sets be A and B respectively.
- Your task it to output a partition that minimize $|A - B|$.
- $n \leq 100000$.

Statistics



Brute Force

- Try all $\binom{n}{a} \leq 2^n$ possible partitions
- Basic recursion
- Fits in time limit for $n \leq 15$
- Expected Score: 25

Solution

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Case 2: $1 + 2 + \ldots + a < S/2$

We claim that if S is even, the optimal $|A - B|$ is 0. If S is odd, the optimal $|A - B|$ is 1. This means we can always find a perfect partition.

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Phase 1

- Step 1: First assign $\{1, 2, \ldots, a\}$ to α (As always, β take the rest).
- Step 2: Let the current $\alpha = \{x, x+1, x+2, \ldots, y-1, y\}$.
- Step 3a: If $A < B$, replace x by $y+1$. Go back to step 2.
- Step 3b: If $A \geq B$, this phase is done.

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First, the procedure must terminate since $a \geq b$, taking the largest a integers from $[n]$ must obtain a greater sum. Second, in each step, we increase A by a, therefore when terminated, $A - B < a$.

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Phase 2

- Let $d = A - B$.
- We would like to find a x in α, and y outside α such that $x - y = \lfloor d/2 \rfloor$.
- Replacing x by y yields the optimal solution.
- Since the largest element in α is at least $a + 1$, we can always find such x, y.