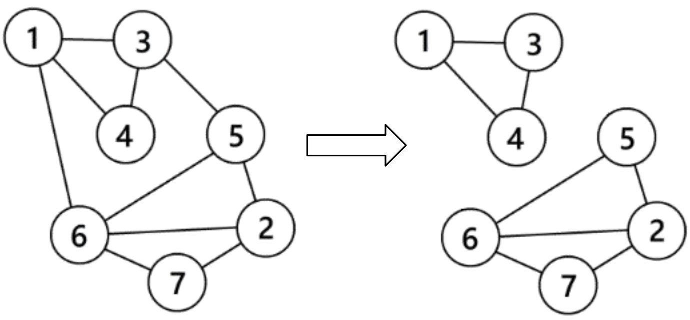
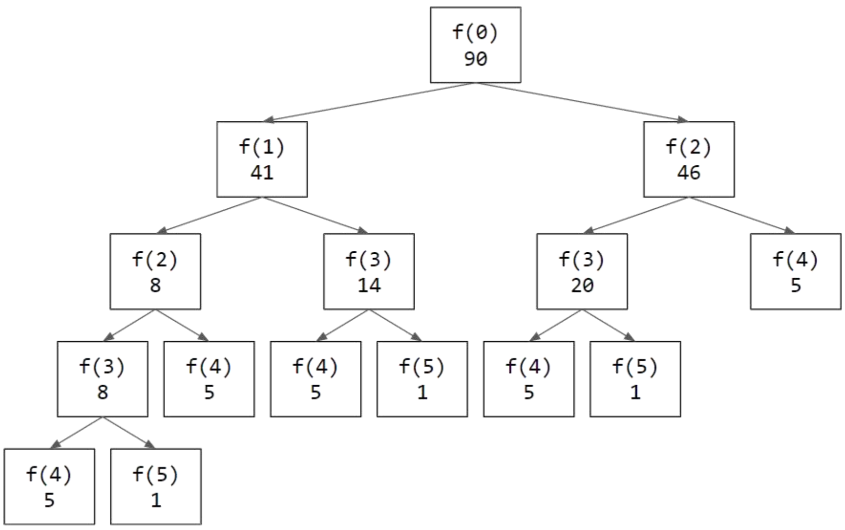
Hong Kong Olympiad in Informatics 2021/22
Heat Event (Senior Group)
Official Solution

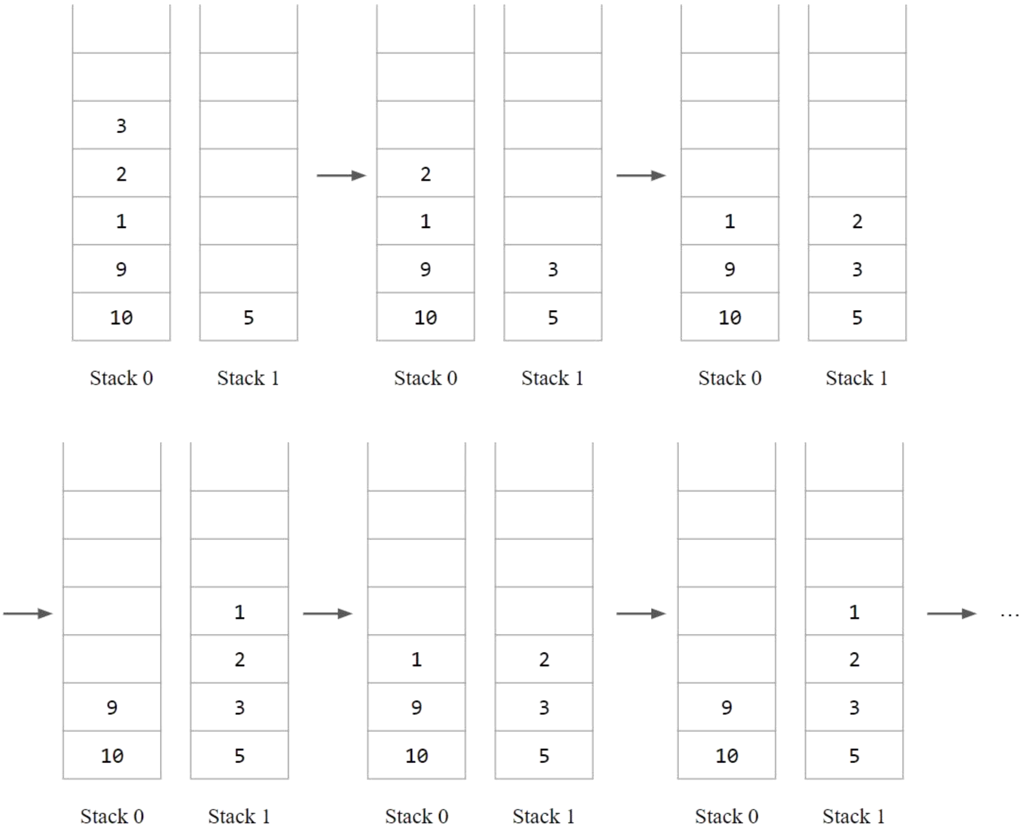
Statistics (N = 264)
Full mark = 45. Maximum = 37.5. Median = 10. Advance to Final = 13.5 marks or above.

Section A

Q	A	Explanation
1	C	i. True. If a word contains 'e', the word must contain 'te' or 'es'. Therefore, words containing 'e' must be more than or equal to words containing 'te' or 'es'. It is easy to find a word that contains 'e' but does not containing 'te' or 'es', such as the word 'the'. Thus the number of words containing 'e' will be strictly larger. ii. False. If the word ends with 'ing', it must have 'n' as the second last letter. Therefore, the number of words ending with 'ing' must be less than or equal to the number of words with 'n' as the second last letter.
2	A	It suffices to trace the program directly. The program counts the edit distance between the string and the string reversed, divided by 2.
3	A	The following figure shows a possible deletion, which involves 2 edges only: 
4	D	i. True. The statement "If Charlie does not study at home, he fails the test." is equivalent to "If Charlie does not fail the test, he has studied at home". As Charlie does not fail the test, Charlie must have studied at home. ii. True. Although Charlie studies at home, it is not necessary that Charlie does not go cycling. Therefore, it may be possible that Charlie has gone cycling. iii. True. It is impossible to deduce whether it is sunny on Saturday from the given information.
5	D	`NOT (((NOT A) AND B) AND C)` ≡ `NOT((NOT A) AND B) OR NOT C` (by De Morgan's Law) ≡ `(A OR NOT B) OR NOT C` (by De Morgan's Law) ≡ `A OR (NOT B OR NOT C)` (by commutative law) When `A ≡ TRUE`, `B ≡ TRUE`, `C ≡ FALSE`, option D gives `FALSE` but option A, B and C gives `TRUE`.

Q	A	Explanation
6	B	Group three or four consecutive digits (before and after the decimal point) of the binary representation 1011101.11011001_2 to convert it to octal and hexadecimal representation respectively. Therefore, $\underline{1}\ \underline{011}\ \underline{101}.\underline{110}\ \underline{110}\ \underline{01(0)}_2 = 135.662_8$ and $\underline{101}\ \underline{1101}.\underline{1101}\ \underline{1001}_2 = 5D.D9_{16}$. Also, $5D.D9_{16} = 5\times16^1+13\times16^0+13\times16^{-1}+9\times16^{-2}=93\frac{217}{256}\neq 93.85$.
7	A	The program counts the number of subarrays with even sum. Consider the prefix sum array p, which is `{0, 3, 4, 8, 9, 14, 23, 25, 31, 36, 39}`. The sum of an subarray is $p_r - p_{l-1}$. Case 1: Both both p_r and p_{l-1} are even. The number of such subarrays $=\frac{5\times(5-1)}{2}=10$. Case 2: Both both p_r and p_{l-1} are odd. The number of such subarrays $=\frac{6\times(6-1)}{2}=15$. The total number of subarrays = 10 + 15 = 25.
8	D	Trace the program carefully. Remember that array `a` is a global variable and might be changed in the recursive function calls. The following figure shows the return values of the function calls induced: 
9	C	i. True. Merge sort only compares elements between subarrays while merging two sorted subarrays into one sorted subarray. As each pair of elements will be merged once only, they are never compared more than once. ii. True.
10	A	This algorithm is a non-standard insertion sort implementation. The subarray `a[0]…a[i]` must be sorted in ascending order after the `i`-th iteration.

Q	A	Explanation
11	B	The path must be in the form $1 \to 3 \to 4 \to 5 \to \ldots \to 5 \to 6 \to 7 \to \ldots \to 7 \to 2$, where the omitted parts are zero or more cycles (either $5 \to 1 \to 3 \to 4 \to 5$ or $7 \to 8 \to 9 \to 6 \to 7$, both of length 4). Therefore, number of edges passed must be in the form of $6 + 4x$, where x is a non-negative integer (in other words, it must be congruent to 2 modulo 4).
12	C	Let p be the probability that Alice wins, we know that p is the sum of the following two parts: 1. Alice wins in the first round: The probability is $\frac{1}{2}$. 2. Alice does not win in the first round but wins subsequently: Alice must fail to get a head and Bob must fail to get two heads. After that, the game "restarts". Hence, the probability is $\frac{1}{2} \times \left(1 - \left(\frac{1}{2}\right)^2\right) \times p = \frac{3}{8}p$. Solving $p = \frac{1}{2} + \frac{3}{8}p$, we can obtain $p = \frac{4}{5}$.
13	D	i. False. If `f(x)` returns `true`, since the expression `f(x) or not f(x)` (Pascal) / `f(x) || !f(x)` (C/C++) must be `true`, the evaluation of the part `not f(x)` (Pascal) / `!f(x)` (C/C++) will be skipped due to short-circuit evaluation. Hence, it is possible that `f(x)` is called once only. ii. False. If `f(x)` is non-deterministic (e.g. if it includes randomness), it may return true in the first call and return false in the second call. `g(x)` will return `false` in such case.

Q	A	Explanation
14	A	The following shows the first few iterations of the for loop:  As the state of the stack repeats with a cycle of length 2, the state of the stack only depends on the parity of the number of iterations. The state after the 2022^{th} iteration would be same as that after the 4^{th} iteration.
15	B	Regardless of the initial content of A, - there is 1 value for `i` such that `aux` is called 1 time only (`A[7]`). - there are 2 values for `i` such that `aux` is called 2 times only (`A[3]` and `A[11]`). - there are 4 values for `i` such that `aux` is called 3 times only (`A[1]`, `A[5]`, `A[9]`, `A[13]`). - `aux` is called 4 times for other values of `i`. The total number of calls = $1 \times 1 + 2 \times 2 + 4 \times 3 + 93 \times 4 = 389$
16	B	Let $x_1, x_2, \ldots, x_k$ be the length of the cycles formed ($x_1 + x_2 + \ldots + x_k \leq 10$). Then $N = \text{LCM}(x_1, x_2, \ldots, x_k)$. $N = 24, 32, 36$ are impossible to achieve, while 30 is achievable by $x = \{2, 3, 5\}$ (for example, by $a = \{2, 1, 4, 5, 3, 7, 8, 9, 10, 6\}$).
17	D	If the graph contains exactly one cycle of length 3, then there are $C_3^4 = 4$ such cycles. The unselected node can be connected to either of the nodes in the cycle (or neither of them), so each cycle corresponds to 4 possible graphs. If the graph contains exactly one cycle of length 4, then node 1 can be opposite to either of nodes 2, 3 and 4. Thus, there are 3 valid graphs. The total number of valid graphs is $4 \times 4 + 3 = 19$ out of all $2^6 = 64$ possibilities.

Q	A	Explanation

18 A

i. True. Note that the probability of rolling the die at least k times is $\left(\frac{11}{12}\right)^{k-1}$ (since it requires the previous $k-1$ rolls to be unsuccessful).

Hence, the expected number of rolls $=1+\frac{11}{12}+\left(\frac{11}{12}\right)^2+\left(\frac{11}{12}\right)^3+\cdots=\frac{1}{1-\frac{11}{12}}=12$.

ii. False. The probability that the face numbered 1 appeared at least once $=1-\left(\frac{11}{12}\right)^6$ $=0.4067\ldots\neq 0.5$.

19 C

The array `a` is `{2, 1, 1, 0, 0}`, with 5 entries. `y` is identical to the number of entries in array `a`.

20 B

The array `a` consists of a `7`, 7 copies of `6`, 7 × 6 copies of `5`, 7 × 6 × 5 copies of `4`, …, 7 × 6 × 5 × 4 × 3 × 2 × 1 copies of `0`.

Total number of entries = 1 + 7 + 7 × 6 + 7 × 6 × 5 + … + 7 × 6 × 5 × 4 × 3 × 2 × 1 = 13700.

21 C

Let `w[i]` be the number of ways to represent the number `i`.

Then, `w[0] = 1`, and `w[i] = w[i-1] + w[i-2] + w[i-4] + w[i-8]`.

`i`	0	1	2	3	4	5	6	7	8	9	10
`w[i]`	1	1	2	3	6	10	18	31	56	98	174

22 C

i. True. In any undirected graph, the sum of degrees of all nodes must be even.

ii. True. Let n be the number of nodes with degree 4. Since each node with degree 4 has exactly 3 children and each node with degree 1 has no children (except the root which has 1 more child), the number of nodes = 1 + 3n + 1 = 3n + 2. n must be a non-negative integer, so the number of nodes is never a multiple of 3.

23 C

The shaded cells in the following table shows the values added to the variable `c` and the `x,y` pairs in each iteration:

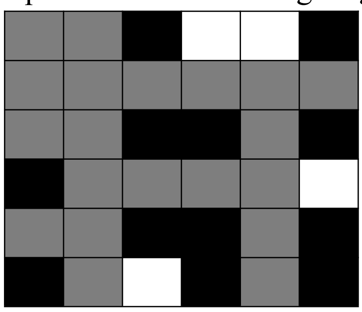
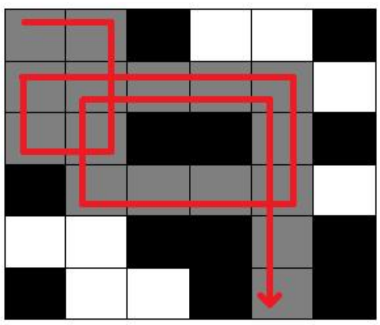
`x` \ `y`	`0`	`1`	`2`	`3`	`4`
`0`	**14**	**9**	21	1	9
`1`	3	**2**	8	19	6
`2`	7	**0**	**16**	**15**	30
`3`	4	27	31	**10**	11
`4`	3	4	5	**6**	**7**

24 C

It is possible to place 14 bishops in the first column and the last column. It is not possible to place 15 bishops since there must be at most 1 bishop in each of the 14 diagonals.

Q	A	Explanation
25	C	$\begin{aligned} f(32) &= f(16) \times 2 + 16^2 \\ &= f(8) \times 4 + 8^2 \times 2 + 16^2 \\ &= f(4) \times 8 + 4^2 \times 4 + 8^2 \times 2 + 16^2 \\ &= f(2) \times 16 + 2^2 \times 8 + 4^2 \times 4 + 8^2 \times 2 + 16^2 \\ &= f(1) \times 32 + 1^2 \times 16 + 2^2 \times 8 + 4^2 \times 4 + 8^2 \times 2 + 16^2 \\ &= 32 + 16 + 32 + 64 + 128 + 256 \\ &= 528 \end{aligned}$

Section B

<table>
<tr><th colspan="4">Answer and Explanation</th></tr>
<tr><th></th><th>Pascal</th><th>C</th><th>C++</th></tr>
<tr><td>A1</td><td><code>c</code></td><td><code>c</code></td><td><code>c</code></td></tr>
<tr><td>A2</td><td><code>c</code></td><td><code>c</code></td><td><code>c</code></td></tr>
<tr><td rowspan="2">A3</td><td><code>1</code></td><td><code>1</code></td><td><code>1</code></td></tr>
<tr><td colspan="3">The function is defined such that in all function calls, $a^b \times c$ remains constant. Therefore in the initial call $c = 1$, and when $b = 0$, c would store the required answer.</td></tr>
<tr><td rowspan="2">B</td><td colspan="3"><code>20</code></td></tr>
<tr><td colspan="3">The shaded (grey) squares in the following diagram are reachable
</td></tr>
<tr><td rowspan="2">C</td><td colspan="3"><code>16</code></td></tr>
<tr><td colspan="3">The shaded (grey) squares in the following diagram are reached in the optimal path (also drawn)
</td></tr>
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<table>
<tr><th colspan="4">Answer and Explanation</th></tr>
<tr><td rowspan="2">D</td><td><code>x-y=i-j</code></td><td><code>x-y==i-j</code></td><td><code>x-y==i-j</code></td></tr>
<tr><td colspan="3">All coordinates that can be hit by the s-attack has the same value of $x - y$.</td></tr>
<tr><td rowspan="2">E</td><td><code>x+y=i+j</code></td><td><code>x+y==i+j</code></td><td><code>x+y==i+j</code></td></tr>
<tr><td colspan="3">All coordinates that can be hit by the e-attack has the same value of $x + y$.</td></tr>
<tr><td>F</td><td colspan="3"><code>3</code></td></tr>
<tr><td rowspan="2">G</td><td colspan="3"><code>6</code></td></tr>
<tr><td colspan="3">We prove the fact that regardless of which number remains in Alice’s array, Peter will always have the closest element to it.
Let Alice’s array be a and Peter’s array be b. Define set s as a subset of b such that each element in s is the closest to some element in a among all elements in b (choose one arbitrarily in case of two closest items).
Obviously, $|a| \geq |s|$ as each element in a contributes to at most 1 element in s.
Since Alice moves first, when it is Peter’s turn, $|b| > |a| \geq |s|$, which implies the existence of an element that is in b but not in s. Peter can just remove such an element.
In the end, $|a| = |b| = |s| = 1$ and Peter’s last element must be the closest to Alice’s last element.
Therefore Alice will choose to leave behind the element whose closest in b is furthest. We check all items in a to get the result.</td></tr>
<tr><td rowspan="2">H</td><td colspan="3"><code>16168</code></td></tr>
<tr><td colspan="3">The return value is equal to 2021 * number of 1 bits in the binary representation of 2021</td></tr>
<tr><td rowspan="2">I</td><td><code>21, ;y:=y+y end;</code></td><td colspan="2"><code>48, y=y+y;} //</code>
<code>47, x=x/2;y=y+y; //</code>
<code>46, ;ans+=y;x+=x-2; //</code>
<code>46 ans+=y;y=y+y;</code></td></tr>
<tr><td colspan="3">The value of y has to be doubled after each iteration, see examples below:
$$211 \times 34 = 34 + \left\lfloor \frac{211}{2} \right\rfloor \times (34 \times 2)$$
$$34 \times 211 = 0 + \left\lfloor \frac{34}{2} \right\rfloor \times (211 \times 2)$$
Where the part to the left of the + sign is added to the <code>ans</code> variable, while the part to the right of the + sign is a reduced version of the problem with new values of x and y.</td></tr>
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Answer and Explanation		

J	`(f()*100+f())div 125 //`
	`f()div 25*20+f()div 5 //`
	`f()div 5*4+f()div 25`
		`(f()*100+f())/125 //`
		`f()/25*20+f()/5 //`
		`f()/5*4+f()/25`

	Or any other reasonable answers	
	For the first solution, we first generate a random number between 0 and 999, then divide the result by 125 to get a random number between 0 and 79 For the second solution, we first generate a random number between 0 and 3, then multiply this by 20, getting 0, 20, 40 or 60, we then add a random number between 0 and 19 to this to get the final result of a random number between 0 and 79. The third solution is similar to the second.	
K	`225`	
	Let $d(x)$ denote the digit sum of the number x, it is well known that $d(x) \equiv x \pmod 9$. We can also see that $f(x) = f(d(x))$ and can therefore deduce that $f(x) = x \pmod 9$. Therefore we get $f(x) = 1$ if and only if $x \equiv 1 \pmod 9$.	
L	`1`	
	The program first computes the first 20 rows of the Pascal's triangle, where `a[i][j]` stores the value of $\binom{i}{j}$, then adds up the last 19 terms on the 20th line: $\binom{19}{1} + \binom{19}{2} + \ldots + \binom{19}{19}$ Using a property of the Pascal's triangle, we know that $\binom{19}{0} + \binom{19}{1} + \binom{19}{2} + \ldots + \binom{19}{19} = 2^{19}$, so $\binom{19}{1} + \binom{19}{2} + \ldots + \binom{19}{19} = 2^{19} - 1$. To find $2^{19} (\mathrm{mod} 9)$, we use Fermat's theorem which states that $a^{\varphi(n)} \equiv 1 \pmod n$, or in this case, $2^6 \equiv 1 \pmod 9$ and therefore $2^{18} \equiv 1 \pmod 9$ and $2^{19} \equiv 2 \pmod 9$. Combining all the steps, we get the following equation to calculate the result: $\binom{19}{1} + \binom{19}{2} + \ldots + \binom{19}{19} = 2^{19} - 1 \equiv 2 - 1 = 1 \pmod 9$	

<table>
<tr><th colspan="2">Answer and Explanation</th></tr>
<tr><td rowspan="2">M</td><td>1</td></tr>
<tr><td>

$$P(\text{first person passes}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(\text{second person does not pass}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(\text{first person does not pass but second person passes}) = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$$

$$E(\text{number of winning participants}) = 2 \times \frac{1}{4} + 1 \times \frac{1}{2} = 1$$

</td></tr>
<tr><td rowspan="2">N</td><td>769/256</td></tr>
<tr><td>

$$P(m \text{ people dies}) = \frac{\binom{8}{m}}{2^8}$$

This is due to there being 8 possible positions to die on, so a total of $\binom{8}{m}$ ways for m people to die. While the first step in each row relies on probability and has 2^8 ways in total

$$E(\text{number of winning participants}) = \frac{\sum_{m=0}^{6} (7-m) \times \binom{8}{m}}{2^8} = \frac{769}{256}$$

</td></tr>
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